Exercise 9

Use the *Laplace transform method* to solve the Volterra integral equations of the first kind:

$$1 + x - \frac{1}{3!}x^3 - e^x = \int_0^x (t - x)u(t)\,dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)\,dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L}\left\{\int_0^x f(x-t)g(t)\,dt\right\}$$

Multiply both sides of the integral equation by -1.

$$\frac{1}{3!}x^3 + e^x - 1 - x = \int_0^x (x - t)u(t)\,dt$$

Take the Laplace transform of both sides of the integral equation.

$$\mathcal{L}\left\{\frac{1}{6}x^3 + e^x - 1 - x\right\} = \mathcal{L}\left\{\int_0^x (x - t)u(t)\,dt\right\}$$

Use the fact that the Laplace transform is linear on the left side and apply the convolution theorem on the right side.

$$\begin{gathered}
\frac{1}{6}\mathcal{L}\{x^3\} + \mathcal{L}\{e^x\} - \mathcal{L}\{1\} - \mathcal{L}\{x\} = \mathcal{L}\{x\}U(s) \\
\frac{1}{6}\left(\frac{6}{s^4}\right) + \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2} = \frac{1}{s^2}U(s)
\end{gathered}$$

Solve for $U(s)$.

$$\begin{aligned}
U(s) &= \frac{1}{s^2} + \frac{s^2}{s-1} - s - 1 \\
&= \frac{1}{s^2} + \frac{s^2 - (s+1)(s-1)}{s-1} \\
&= \frac{1}{s^2} + \frac{1}{s-1}
\end{aligned}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$\begin{aligned}
u(x) &= \mathcal{L}^{-1}\{U(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \\
&= x + e^x
\end{aligned}$$