Exercise 1.1.1

(a) Prove that if $\lim_{n\to\infty} n^p u_n = A < \infty$, $p > 1$, the series $\sum_{n=1}^{\infty} u_n$ converges.

(b) Prove that if $\lim_{n\to\infty} n u_n = A > 0$, the series diverges. (The test fails for $A = 0$.) These two tests, known as **limit tests**, are often convenient for establishing the convergence of a series. They may be treated as comparison tests, comparing with

$$\sum_n n^{-q}, \quad 1 \leq q < p.$$

Solution

Part (a)

Suppose that

$$\lim_{n\to\infty} n^p u_n = A,$$

where A is finite. There are many possible formulas for u_n, for example,

$$u_n = \frac{A\cos^2\left(\frac{1}{n}\right)}{n^p}.$$

However, the highest it can be (the upper bound) is

$$u_n = \frac{A}{n^p};$$

otherwise, A will be infinite. If $\sum_{n=1}^{\infty} u_n$ converges using the upper bound, then it will converge using any formula with values less than it.

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{A}{n^p} = A\sum_{n=1}^{\infty} n^{-p} \tag{1}$$

Since $f(n) = n^{-p}$ is continuous, positive, and decreasing on the interval $1 \leq n < \infty$, the Integral Test can be applied.

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} x^{-p}\,dx = -\frac{1}{p}x^{-p+1}\bigg|_1^{\infty} = \frac{1}{p}$$

This integral converges to $1/p$ because $p > 1$. Therefore,

$$\sum_{n=1}^{\infty} u_n$$

converges by the Integral Test. The series in equation (1) is known as the p-series.

Part (b)

Suppose that

$$\lim_{n\to\infty} nu_n = A,$$

where $A > 0$. There are many possible formulas for u_n, for example,

$$u_n = n.$$

However, the lowest it can be (the lower bound) is

$$u_n = \frac{A}{n};$$

any lower and A will be zero or less. If $\sum_{n=1}^{\infty} u_n$ diverges using the lower bound, then it will diverge using any formula with values greater than it.

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{A}{n} = A \sum_{n=1}^{\infty} n^{-1} \tag{2}$$

Since $f(n) = n^{-1}$ is continuous, positive, and decreasing on the interval $1 \leq n < \infty$, the Integral Test can be applied.

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} x^{-1}\,dx = \ln x\Big|_1^{\infty} = \ln\infty - \ln 1 = \infty$$

This integral diverges because the natural logarithm increases indefinitely. Therefore,

$$\sum_{n=1}^{\infty} u_n$$

diverges by the Integral Test. The series in equation (2) is known as the harmonic series.