Exercise 1.1.5

Test for convergence

(a) $\sum_{n=2}^{\infty}(\ln n)^{-1}$

(b) $\sum_{n=1}^{\infty}\frac{n!}{10^n}$

(c) $\sum_{n=1}^{\infty}\frac{1}{2n(2n+1)}$

(d) $\sum_{n=1}^{\infty}[n(n+1)]^{-1/2}$

(e) $\sum_{n=0}^{\infty}\frac{1}{2n+1}$

Solution

Part (a)

Use the direct comparison test with the harmonic series.

$$\sum_{n=2}^{\infty}(\ln n)^{-1} = \sum_{n=2}^{\infty}\frac{1}{\ln n} > \underbrace{\sum_{n=2}^{\infty}\frac{1}{n}}_{\text{diverges}} \quad \Rightarrow \quad \sum_{n=2}^{\infty}(\ln n)^{-1} \quad \text{diverges}$$

Part (b)

Since exponents and factorials are involved, use the ratio test.

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{(n+1)!}{10^{n+1}}}{\frac{n!}{10^n}} = \lim_{n\to\infty}\frac{(n+1)!}{10^{n+1}}\times\frac{10^n}{n!} = \lim_{n\to\infty}\frac{n+1}{10} = \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty}\frac{n!}{10^n} \quad \text{diverges}$$

Part (c)

Use the direct comparison test with the p-series.

$$\sum_{n=1}^{\infty}\frac{1}{2n(2n+1)} = \sum_{n=1}^{\infty}\frac{1}{4n^2+2n} < \underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2}}_{\text{converges}} \quad \Rightarrow \quad \sum_{n=1}^{\infty}\frac{1}{2n(2n+1)} \quad \text{converges}$$

Part (d)

Use the direct comparison test with the harmonic series.

$$\sum_{n=1}^{\infty}[n(n+1)]^{-1/2} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}} > \sum_{n=1}^{\infty}\frac{1}{\sqrt{(n+1)(n+1)}}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n+1}$$

$$= \underbrace{\sum_{n=2}^{\infty}\frac{1}{n}}_{\text{diverges}} \quad \Rightarrow \quad \sum_{n=1}^{\infty}[n(n+1)]^{-1/2} \quad \text{diverges}$$

Part (e)

Use the direct comparison test with the harmonic series.

$$\sum_{n=0}^{\infty}\frac{1}{2n+1} > \sum_{n=0}^{\infty}\frac{1}{2n+2}$$

$$= \sum_{n=0}^{\infty}\frac{1}{2(n+1)}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n+1}$$

$$= \frac{1}{2}\underbrace{\sum_{n=1}^{\infty}\frac{1}{n}}_{\text{diverges}} \quad \Rightarrow \quad \sum_{n=0}^{\infty}\frac{1}{2n+1} \quad \text{diverges}$$