Exercise 1.1.7

For what values of p and q will $\sum_{n=2}^{\infty} \frac{1}{n^p (\ln n)^q}$ converge?

$$\textit{ANS.}\quad \text{Convergent for } \begin{cases} p > 1, & \text{all } q, \\ p = 1, & q > 1, \end{cases} \quad \text{divergent for } \begin{cases} p < 1, & \text{all } q, \\ p = 1, & q \leq 1. \end{cases}$$

Solution

The summand can be integrated, so use the integral test. Let

$$f(x) = \frac{1}{x^p (\ln x)^q}.$$

x and $\ln x$ are both continuous functions for $x \geq 2$, so their product $x^p (\ln x)^q$ is continuous on this interval. So is their reciprocal $1/[x^p(\ln x)^q]$. $f(x)$ is positive for $x \geq 2$. Calculate the first derivative of $f(x)$.

$$f'(x) = \frac{d}{dx}\left[\frac{1}{x^p (\ln x)^q}\right] = -\frac{q + p \ln x}{x^{p+1} (\ln x)^{q+1}}$$

Provided that p and q are positive numbers, $f'(x) < 0$ for all $x \geq 2$, so $f(x)$ is a monotonically decreasing function. The conditions for using the integral test are satisfied; now evaluate the corresponding integral by using the substitution $u = \ln x$ ($du = dx/x$).

$$\begin{aligned} \int_2^\infty \frac{dx}{x^p (\ln x)^q} = \int_{\ln 2}^\infty \frac{du}{x^{p-1} u^q} &= \int_{\ln 2}^\infty \frac{du}{(e^u)^{p-1} u^q} \\ &= \int_{\ln 2}^\infty \frac{du}{e^{(p-1)u} u^q} \\ &= \int_{\ln 2}^\infty e^{-(p-1)u} u^{-q}\, du \end{aligned}$$

In order for this integral to converge, either $p - 1 > 0$ (in which case q can be any positive number) or $p - 1 = 0$ (in which case the integrand corresponds to the p-series, which sets the condition $q > 1$ for convergence).