Exercise 10.1.1

Show that

$$G(x,t) = \begin{cases} x, & 0 \le x < t, \\ t, & t < x \le 1, \end{cases}$$

is the Green's function for the operator $\mathcal{L} = -d^2/dx^2$ and the boundary conditions $y(0) = 0$, $y'(1) = 0$.

Solution

The Green's function for an operator $\mathcal{L}$ satisfies

$$\mathcal{L}G = \delta(x-t).$$

For the operator $\mathcal{L} = -d^2/dx^2$, this equation becomes

$$-\frac{d^2G}{dx^2} = \delta(x-t). \tag{1}$$

If $x \neq t$, then the right side is zero.

$$-\frac{d^2G}{dx^2} = 0, \quad x \neq t$$

The general solution is obtained by integrating both sides with respect to x twice. Different constants are needed for $x < t$ and for $x > t$.

$$G(x,t) = \begin{cases} C_1x + C_2 & \text{if } 0 \le x < t \\ C_3x + C_4 & \text{if } t < x \le 1 \end{cases}$$

Four conditions are needed to determine these four constants. Two of them are obtained from the provided boundary conditions.

$$G(0,t) = C_1(0) + C_2 = 0 \quad \rightarrow \quad C_2 = 0$$
$$\frac{dG}{dx}(1,t) = C_3 = 0$$

As a result, the Green's function becomes

$$G(x,t) = \begin{cases} C_1x & \text{if } 0 \le x < t \\ C_4 & \text{if } t < x \le 1 \end{cases}.$$

The third condition comes from the fact that the Green's function must be continuous at $x = t$: $G(t-,t) = G(t+,t)$.

$$C_1(t) = C_4$$

Consequently, the Green's function becomes

$$G(x,t) = \begin{cases} C_1x & \text{if } 0 \le x < t \\ C_1t & \text{if } t < x \le 1 \end{cases}.$$

The fourth and final condition is obtained from the defining equation of the Green's function, equation (1).

$$-\frac{d^2G}{dx^2} = \delta(x-t)$$

Integrate both sides with respect to x from $t-$ to $t+$.

$$-\int_{t-}^{t+} \frac{d^2G}{dx^2}\,dx = \int_{t-}^{t+} \delta(x-t)\,dx$$

$$-\frac{dG}{dx}\bigg|_{t-}^{t+} = 1$$

$$-\frac{dG}{dx}(t+,t) + \frac{dG}{dx}(t-,t) = 1$$

$$-(0) + (C_1) = 1$$

$$C_1 = 1$$

Therefore, the Green's function for $\mathcal{L} = -d^2/dx^2$ subject to the provided boundary conditions is

$$G(x,t) = \begin{cases} x & \text{if } 0 \leq x < t \\ t & \text{if } t < x \leq 1 \end{cases}.$$