Exercise 10.1.2

Find the Green's function for

$$\text{(a)} \quad \mathcal{L}y(x) = \frac{d^2y(x)}{dx^2} + y(x), \qquad \begin{cases} y(0) = 0, \\ y'(1) = 0. \end{cases}$$

$$\text{(b)} \quad \mathcal{L}y(x) = \frac{d^2y(x)}{dx^2} - y(x), \quad y(x) \text{ finite for } -\infty < x < \infty.$$

Solution

The Green's function for an operator $\mathcal{L}$ satisfies

$$\mathcal{L}G = \delta(x-t).$$

Part (a)

For the operator $\mathcal{L} = d^2/dx^2 + 1$, this equation becomes

$$\frac{d^2G}{dx^2} + G = \delta(x-t). \tag{1}$$

If $x \neq t$, then the right side is zero.

$$\frac{d^2G}{dx^2} + G = 0, \quad x \neq t$$

The general solution can be written in terms of sine and cosine. Different constants are needed for $x < t$ and for $x > t$.

$$G(x,t) = \begin{cases} C_1 \cos x + C_2 \sin x & \text{if } 0 \leq x < t \\ C_3 \cos x + C_4 \sin x & \text{if } t < x \leq 1 \end{cases}$$

Four conditions are needed to determine these four constants. Two of them are obtained from the provided boundary conditions.

$$\begin{aligned} G(0,t) = C_1(1) + C_2(0) = 0 \qquad &\rightarrow \qquad C_1 = 0 \\ \frac{dG}{dx}(1,t) = -C_3 \sin 1 + C_4 \cos 1 = 0 \qquad &\rightarrow \qquad C_4 = C_3 \frac{\sin 1}{\cos 1} = C_3 \tan 1 \end{aligned}$$

As a result, the Green's function becomes

$$G(x,t) = \begin{cases} C_2 \sin x & \text{if } 0 \leq x < t \\ C_3 \cos x + (C_3 \tan 1) \sin x & \text{if } t < x \leq 1 \end{cases}.$$

The third condition comes from the fact that the Green's function must be continuous at $x = t$: $G(t-,t) = G(t+,t)$.

$$C_2 \sin t = C_3 \cos t + (C_3 \tan 1) \sin t$$

Divide both sides by $\cos t$.

$$C_2 \tan t = C_3 + (C_3 \tan 1) \tan t$$

Solve for C_3.

$$C_3 = C_2 \frac{\tan t}{1 + \tan 1 \tan t} \tag{2}$$

The fourth and final condition is obtained from the defining equation of the Green's function, equation (1).

$$\frac{d^2G}{dx^2} + G = \delta(x - t)$$

Integrate both sides with respect to x from $t-$ to $t+$.

$$\int_{t-}^{t+} \left(\frac{d^2G}{dx^2} + G\right) dx = \int_{t-}^{t+} \delta(x - t)\, dx$$

$$\int_{t-}^{t+} \frac{d^2G}{dx^2}\, dx + \underbrace{\int_{t-}^{t+} G\, dx}_{=0} = \underbrace{\int_{t-}^{t+} \delta(x - t)\, dx}_{=1}$$

$$\left.\frac{dG}{dx}\right|_{t-}^{t+} = 1$$

$$\frac{dG}{dx}(t+, t) - \frac{dG}{dx}(t-, t) = 1$$

$$(-C_3 \sin t + C_3 \tan 1 \cos t) - (C_2 \cos t) = 1$$

Divide both sides by $\cos t$.

$$-C_3 \tan t + C_3 \tan 1 - C_2 = \frac{1}{\cos t}$$

$$C_3(\tan 1 - \tan t) = \frac{1}{\cos t} + C_2$$

Divide both sides by $\tan 1 - \tan t$.

$$C_3 = \frac{1}{\cos t(\tan 1 - \tan t)} + \frac{C_2}{\tan 1 - \tan t}$$

Substitute equation (2) for C_3.

$$C_2 \frac{\tan t}{1 + \tan 1 \tan t} = \frac{1}{\cos t(\tan 1 - \tan t)} + \frac{C_2}{\tan 1 - \tan t}$$

Solve for C_2.

$$C_2 \left(\frac{\tan t}{1 + \tan 1 \tan t} - \frac{1}{\tan 1 - \tan t}\right) = \frac{1}{\cos t(\tan 1 - \tan t)}$$

$$-C_2 \frac{\tan^2 t + 1}{(1 + \tan 1 \tan t)(\tan 1 - \tan t)} = \frac{1}{\cos t(\tan 1 - \tan t)}$$

$$-C_2 \frac{\sec^2 t}{1 + \tan 1 \tan t} = \sec t$$

$$C_2 = -\cos t(1 + \tan 1 \tan t)$$

Use equation (2) to get C_3.

$$C_3 = C_2 \frac{\tan t}{1 + \tan 1 \tan t} = (-\cos t)(\tan t) = -\sin t$$

Therefore, the Green's function for $\mathcal{L} = d^2/dx^2 + 1$ subject to the provided boundary conditions is

$$G(x, t) = \begin{cases} -\cos t(1 + \tan 1 \tan t) \sin x & \text{if } 0 \le x < t \\ -\sin t \cos x + (-\sin t \tan 1) \sin x & \text{if } t < x \le 1 \end{cases}.$$

Part (b)

For the operator $\mathcal{L} = d^2/dx^2 - 1$, this equation becomes

$$\frac{d^2G}{dx^2} - G = \delta(x - t). \tag{3}$$

If $x \neq t$, then the right side is zero.

$$\frac{d^2G}{dx^2} - G = 0, \quad x \neq t$$

The general solution can be written in terms of exponential functions. Different constants are needed for $x < t$ and for $x > t$.

$$G(x,t) = \begin{cases} C_5e^{-x} + C_6e^{x} & \text{if } -\infty < x < t \\ C_7e^{-x} + C_8e^{x} & \text{if } t < x < \infty \end{cases}$$

Four conditions are needed to determine these four constants. Two of them are obtained from the provided boundary condition.

$$y(x) \text{ finite for } -\infty < x < \infty. \quad \Rightarrow \quad \begin{cases} C_5 = 0 \\ C_8 = 0 \end{cases}$$

As a result, the Green's function becomes

$$G(x,t) = \begin{cases} C_6e^{x} & \text{if } -\infty < x < t \\ C_7e^{-x} & \text{if } t < x < \infty \end{cases}.$$

The third condition comes from the fact that the Green's function must be continuous at $x = t$: $G(t-,t) = G(t+,t)$.

$$C_6e^{t} = C_7e^{-t}$$

Solve for C_7.

$$C_7 = C_6e^{2t} \tag{4}$$

The fourth and final condition is obtained from the defining equation of the Green's function, equation (3).

$$\frac{d^2G}{dx^2} - G = \delta(x - t)$$

Integrate both sides with respect to x from $t-$ to $t+$.

$$\int_{t-}^{t+} \left(\frac{d^2G}{dx^2} - G\right) dx = \int_{t-}^{t+} \delta(x - t)\, dx$$

$$\int_{t-}^{t+} \frac{d^2G}{dx^2}\, dx - \underbrace{\int_{t-}^{t+} G\, dx}_{=\,0} = \underbrace{\int_{t-}^{t+} \delta(x - t)\, dx}_{=\,1}$$

$$\left.\frac{dG}{dx}\right|_{t-}^{t+} = 1$$

$$\frac{dG}{dx}(t+,t) - \frac{dG}{dx}(t-,t) = 1$$

$$(-C_7e^{-t}) - (C_6e^t) = 1$$

Substitute equation (4) for C_7.

$$(-C_6e^{2t}e^{-t}) - (C_6e^t) = 1$$

$$-2C_6e^t = 1$$

$$C_6 = -\frac{1}{2}e^{-t}$$

Use equation (4) to get C_7.

$$C_7 = C_6e^{2t} = -\frac{1}{2}e^t$$

Therefore, the Green's function for $\mathcal{L} = d^2/dx^2 - 1$ subject to the provided boundary condition is

$$G(x,t) = \begin{cases} -\frac{1}{2}e^{-t}e^x & \text{if } -\infty < x < t \\ -\frac{1}{2}e^te^{-x} & \text{if } t < x < \infty \end{cases}.$$