Exercise 10.1.4

Find the Green's function for the equation

$$-\frac{d^2y}{dx^2} - \frac{y}{4} = f(x),$$

with boundary conditions $y(0) = y(\pi) = 0$.

$$ANS. \quad G(x,t) = \begin{cases} 2\sin(x/2)\cos(t/2), & 0 \le x < t, \\ 2\cos(x/2)\sin(t/2), & t < x \le \pi. \end{cases}$$

Solution

The Green's function for an operator $\mathcal{L}$ satisfies

$$\mathcal{L}G = \delta(x-t).$$

Part (a)

For the operator $\mathcal{L} = -d^2/dx^2 - 1/4$, this equation becomes

$$-\frac{d^2G}{dx^2} - \frac{G}{4} = \delta(x-t). \tag{1}$$

If $x \neq t$, then the right side is zero.

$$-\frac{d^2G}{dx^2} - \frac{G}{4} = 0, \quad x \neq t$$

The general solution can be written in terms of sine and cosine. Different constants are needed for $x < t$ and for $x > t$.

$$G(x,t) = \begin{cases} C_1\cos(x/2) + C_2\sin(x/2) & \text{if } 0 \le x < t \\ C_3\cos(x/2) + C_4\sin(x/2) & \text{if } t < x \le \pi \end{cases}$$

Four conditions are needed to determine these four constants. Two of them are obtained from the provided boundary conditions.

$$\begin{aligned} G(0,t) &= C_1(1) + C_2(0) = 0 \quad &\rightarrow \quad C_1 = 0 \\ G(\pi,t) &= C_3(0) + C_4(1) = 0 \quad &\rightarrow \quad C_4 = 0 \end{aligned}$$

As a result, the Green's function becomes

$$G(x,t) = \begin{cases} C_2\sin(x/2) & \text{if } 0 \le x < t \\ C_3\cos(x/2) & \text{if } t < x \le \pi \end{cases}.$$

The third condition comes from the fact that the Green's function must be continuous at $x = t$: $G(t-,t) = G(t+,t)$.

$$C_2\sin\frac{t}{2} = C_3\cos\frac{t}{2} \quad \rightarrow \quad C_3 = C_2\frac{\sin\frac{t}{2}}{\cos\frac{t}{2}} \tag{2}$$

The fourth and final condition is obtained from the defining equation of the Green's function, equation (1).

$$-\frac{d^2G}{dx^2} - \frac{G}{4} = \delta(x-t)$$

Integrate both sides with respect to x from $t-$ to $t+$.

$$\int_{t-}^{t+} \left(-\frac{d^2G}{dx^2} - \frac{G}{4}\right) dx = \int_{t-}^{t+} \delta(x-t)\, dx$$

$$-\int_{t-}^{t+} \frac{d^2G}{dx^2}\, dx - \frac{1}{4}\underbrace{\int_{t-}^{t+} G\, dx}_{=0} = \underbrace{\int_{t-}^{t+} \delta(x-t)\, dx}_{=1}$$

$$-\frac{dG}{dx}\bigg|_{t-}^{t+} = 1$$

$$-\frac{dG}{dx}(t+,t) + \frac{dG}{dx}(t-,t) = 1$$

$$\frac{C_3}{2}\sin\frac{t}{2} + \frac{C_2}{2}\cos\frac{t}{2} = 1$$

Substitute equation (2) for C_3.

$$\frac{C_2}{2}\frac{\sin\frac{t}{2}}{\cos\frac{t}{2}}\sin\frac{t}{2} + \frac{C_2}{2}\cos\frac{t}{2} = 1$$

Multiply both sides by $2\cos\frac{t}{2}$.

$$C_2\left(\sin^2\frac{t}{2} + \cos^2\frac{t}{2}\right) = 2\cos\frac{t}{2}$$

$$C_2 = 2\cos\frac{t}{2}$$

Use equation (2) to get C_3.

$$C_3 = C_2\frac{\sin\frac{t}{2}}{\cos\frac{t}{2}} = 2\sin\frac{t}{2}$$

Therefore, the Green's function for $\mathcal{L} = -d^2/dx^2 - 1/4$ subject to the provided boundary conditions is

$$G(x,t) = \begin{cases} 2\cos\frac{t}{2}\sin\frac{x}{2} & \text{if } 0 \le x < t \\ 2\sin\frac{t}{2}\cos\frac{x}{2} & \text{if } t < x \le \pi \end{cases}.$$