Exercise 7.2.11

Show that

$$y(x) = \exp\left[-\int^x p(t)\,dt\right]\left\{\int^x \exp\left[\int^s p(t)\,dt\right] q(s)\,ds + C\right\}$$

is a solution of

$$\frac{dy}{dx} + p(x)y(x) = q(x)$$

by differentiating the expression for $y(x)$ and substituting into the differential equation.

Solution

Take the derivative of $y(x)$ by using the product rule.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left\{\exp\left[-\int^x p(t)\,dt\right]\right\}\left\{\int^x \exp\left[\int^s p(t)\,dt\right] q(s)\,ds + C\right\}\\ &\quad + \exp\left[-\int^x p(t)\,dt\right]\frac{d}{dx}\left\{\int^x \exp\left[\int^s p(t)\,dt\right] q(s)\,ds + C\right\}\\ &= \exp\left[-\int^x p(t)\,dt\right]\frac{d}{dx}\left[-\int^x p(t)\,dt\right]\left\{\int^x \exp\left[\int^s p(t)\,dt\right] q(s)\,ds + C\right\}\\ &\quad + \exp\left[-\int^x p(t)\,dt\right]\left\{\frac{d}{dx}\int^x \exp\left[\int^s p(t)\,dt\right] q(s)\,ds + \frac{d}{dx}(C)\right\}\end{aligned}$$

Apply the fundamental theorem of calculus to differentiate the integrals.

$$\begin{aligned}\frac{dy}{dx} &= \exp\left[-\int^x p(t)\,dt\right][-p(x)]\left\{\int^x \exp\left[\int^s p(t)\,dt\right] q(s)\,ds + C\right\}\\ &\quad + \exp\left[-\int^x p(t)\,dt\right]\left\{\exp\left[\int^x p(t)\,dt\right] q(x) + 0\right\}\\ &= -p(x)\exp\left[-\int^x p(t)\,dt\right]\left\{\int^x \exp\left[\int^s p(t)\,dt\right] q(s)\,ds + C\right\}\\ &\quad + \exp(0)q(x)\end{aligned}$$

Now plug this result and the formula for $y(x)$ into the ODE.

$$\begin{aligned}\frac{dy}{dx} + p(x)y(x) &= -p(x)\exp\left[-\int^x p(t)\,dt\right]\left\{\int^x \exp\left[\int^s p(t)\,dt\right] q(s)\,ds + C\right\}\\ &\quad + \exp(0)q(x) + p(x)\exp\left[-\int^x p(t)\,dt\right]\left\{\int^x \exp\left[\int^s p(t)\,dt\right] q(s)\,ds + C\right\}\\ &= \exp(0)q(x) = q(x)\end{aligned}$$