Exercise 7.2.2

The Laplace transform of Bessel's equation ($n = 0$) leads to

$$(s^2 + 1)f'(s) + sf(s) = 0.$$

Solve for $f(s)$.

Solution

Bring $sf(s)$ to the right side.

$$(s^2 + 1)f'(s) = -sf(s)$$

Divide both sides by $s^2 + 1$.

$$f'(s) = -\frac{s}{s^2 + 1}f(s)$$

Divide both sides by $f(s)$.

$$\frac{f'(s)}{f(s)} = -\frac{s}{s^2 + 1}$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{ds}\ln f(s) = -\frac{s}{s^2 + 1}$$

Integrate both sides with respect to s.

$$\ln f(s) = -\int^s \frac{r}{r^2 + 1}\,dr + C$$

To evaluate the integral, let $u = r^2 + 1$. Then $du = 2r\,dr$.

$$\begin{aligned}
\ln f(s) &= -\int^{s^2+1} \frac{1}{u}\frac{du}{2} + C \\
&= -\frac{1}{2}\int^{s^2+1} \frac{du}{u} + C \\
&= -\frac{1}{2}\ln u\bigg|^{s^2+1} + C \\
&= -\frac{1}{2}\ln(s^2 + 1) + C \\
&= \ln(s^2 + 1)^{-1/2} + C
\end{aligned}$$

Exponentiate both sides.

$$\begin{aligned}
e^{\ln f(s)} &= e^{\ln(s^2+1)^{-1/2}+C} \\
f(s) &= e^{\ln(s^2+1)^{-1/2}}e^C \\
&= (s^2 + 1)^{-1/2}e^C \\
&= \frac{e^C}{(s^2 + 1)^{1/2}}
\end{aligned}$$

Therefore, using a new constant A for e^C,

$$f(s) = \frac{A}{(s^2 + 1)^{1/2}}.$$