Exercise 7.2.5

A boat, coasting through the water, experiences a resisting force proportional to v^n, v being the boat's instantaneous velocity. Newton's second law leads to

$$m\frac{dv}{dt} = -kv^n.$$

With $v(t = 0) = v_0$, $x(t = 0) = 0$, integrate to find v as a function of time and v as a function of distance.

Solution

Solve the ODE by separating variables.

$$\frac{dv}{v^n} = -\frac{k}{m}\,dt$$

Integrate both sides.

$$\int v^{-n}\,dv = \int -\frac{k}{m}\,dt \tag{1}$$

Suppose first that $n \neq 1$.

$$\frac{1}{-n+1}v^{-n+1} = -\frac{k}{m}t + C_1$$

Multiply both sides by $-n+1$, using a new constant C_2 for $C_1(-n+1)$.

$$v^{1-n} = -\frac{k}{m}(1-n)t + C_2$$

Now apply the initial condition $v(0) = v_0$ to determine C_2.

$$v_0^{1-n} = C_2$$

Then the previous equation becomes

$$v^{1-n} = -\frac{k}{m}(1-n)t + v_0^{1-n}.$$

Take the $1-n$ root of both sides.

$$v(t) = \left[-\frac{k}{m}(1-n)t + v_0^{1-n}\right]^{1/(1-n)}$$

Suppose secondly that $n = 1$. Then equation (1) becomes

$$\ln v = -\frac{k}{m}t + C_3.$$

Apply the initial condition $v(0) = v_0$ now to determine C_3.

$$\ln v_0 = C_3$$

Substitute this result into the previous equation.

$$\ln v = -\frac{k}{m}t + \ln v_0$$

Exponentiate both sides.

$$\begin{aligned} v(t) &= e^{-kt/m+\ln v_0} \\ &= e^{-kt/m}e^{\ln v_0} \\ &= v_0 e^{-kt/m} \end{aligned}$$

Therefore, as a function of time, the boat velocity is

$$\boxed{v(t) = \begin{cases} v_0 e^{-kt/m} & n = 1 \\ \left[-\dfrac{k}{m}(1-n)t + v_0^{1-n}\right]^{1/(1-n)} & n \neq 1 \end{cases}}.$$

Use the chain rule to write v in terms of x.

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

Substitute this formula into the ODE.

$$mv\frac{dv}{dx} = -kv^n.$$

Separate variables once again.

$$v^{1-n}\,dv = -\frac{k}{m}\,dx$$

Integrate both sides.

$$\int v^{1-n}\,dv = \int -\frac{k}{m}\,dx \tag{2}$$

Suppose first that $n \neq 2$.

$$\frac{1}{2-n}v^{2-n} = -\frac{k}{m}x + C_3$$

Multiply both sides by $2-n$, using a new constant C_4 for $C_3(2-n)$.

$$v^{2-n} = -\frac{k}{m}(2-n)x + C_4$$

Use the two initial conditions, $v(0) = v_0$ and $x(0) = 0$, to determine C_4.

$$v_0^{2-n} = C_4$$

Then the previous equation becomes

$$v^{2-n} = -\frac{k}{m}(2-n)x + v_0^{2-n}.$$

Take the $2-n$ root of both sides.

$$v(x) = \left[-\frac{k}{m}(2-n)x + v_0^{2-n}\right]^{1/(2-n)}$$

Suppose secondly that $n = 2$. Then equation (2) becomes

$$\ln v = -\frac{k}{m}x + C_5.$$

Use the two initial conditions, $v(0) = v_0$ and $x(0) = 0$, to determine C_5.

$$\ln v_0 = C_5$$

The previous equation becomes

$$\ln v = -\frac{k}{m}x + \ln v_0.$$

Exponentiate both sides.

$$\begin{aligned} v(x) &= e^{-kx/m + \ln v_0} \\ &= e^{-kx/m}e^{\ln v_0} \\ &= v_0 e^{-kx/m} \end{aligned}$$

Therefore, as a function of position, the boat velocity is

$$\boxed{v(x) = \begin{cases} v_0 e^{-kx/m} & n = 2 \\ \left[-\frac{k}{m}(2-n)x + v_0^{2-n}\right]^{1/(2-n)} & n \neq 2 \end{cases}.}$$