Exercise 7.2.8

The differential equation

$$P(x, y)\,dx + Q(x, y)\,dy = 0$$

is **exact**. If

$$\varphi(x, y) = \int_{x_0}^{x} P(x, y)\,dx + \int_{y_0}^{y} Q(x_0, y)\,dy,$$

show that

$$\frac{\partial\varphi}{\partial x} = P(x, y), \quad \frac{\partial\varphi}{\partial y} = Q(x, y).$$

Hence, $\varphi(x, y) =$ constant is a solution of the original differential equation.

Solution

Use different dummy variables of integration.

$$\varphi(x, y) = \int_{x_0}^{x} P(r, y)\,dr + \int_{y_0}^{y} Q(x_0, s)\,ds$$

Take the derivative of both sides with respect to x.

$$\begin{aligned}\frac{\partial\varphi}{\partial x} &= \frac{\partial}{\partial x}\left[\int_{x_0}^{x} P(r, y)\,dr + \int_{y_0}^{y} Q(x_0, s)\,ds\right] \\ &= \frac{\partial}{\partial x}\int_{x_0}^{x} P(r, y)\,dr + \frac{\partial}{\partial x}\int_{y_0}^{y} Q(x_0, s)\,ds\end{aligned}$$

The second integral is independent of x, so its derivative is zero. (x_0 is just a constant.)

$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}\int_{x_0}^{x} P(r, y)\,dr$$

Apply the fundamental theorem of calculus to differentiate the integral.

$$\begin{aligned}\frac{\partial\varphi}{\partial x} &= P(r, y)\Big|_{r=x} \\ &= P(x, y)\end{aligned}$$

Now take the derivative of φ with respect to y.

$$\begin{aligned}\frac{\partial\varphi}{\partial y} &= \frac{\partial}{\partial y}\left[\int_{x_0}^{x} P(r, y)\,dr + \int_{y_0}^{y} Q(x_0, s)\,ds\right] \\ &= \frac{\partial}{\partial y}\int_{x_0}^{x} P(r, y)\,dr + \frac{\partial}{\partial y}\int_{y_0}^{y} Q(x_0, s)\,ds\end{aligned}$$

Apply the fundamental theorem of calculus to differentiate the second integral.

$$\frac{\partial\varphi}{\partial y} = \frac{\partial}{\partial y}\int_{x_0}^{x} P(r,y)\,dr + Q(x_0,s)\bigg|_{s=y}$$
$$= \frac{\partial}{\partial y}\int_{x_0}^{x} P(r,y)\,dr + Q(x_0,y)$$

Because the limits of integration are independent of y, the derivative can be brought inside the integrand by the Leibniz rule.

$$\begin{aligned}
\frac{\partial\varphi}{\partial y} &= \int_{x_0}^{x} \frac{\partial P}{\partial y}\bigg|_{x=r} dr + Q(x_0,y) \\
&= \int_{x_0}^{x} \left[\frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial x}\right)\right]\bigg|_{x=r} dr + Q(x_0,y) \\
&= \int_{x_0}^{x} \left[\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial y}\right)\right]\bigg|_{x=r} dr + Q(x_0,y) \\
&= \frac{\partial\varphi}{\partial y}\bigg|_{x_0}^{x} + Q(x_0,y) \\
&= \frac{\partial\varphi}{\partial y}(x,y) - \frac{\partial\varphi}{\partial y}(x_0,y) + Q(x_0,y)
\end{aligned}$$

This first term on the right cancels the one on the left.

$$0 = -\frac{\partial\varphi}{\partial y}(x_0,y) + Q(x_0,y)$$

Solve for the derivative.

$$\frac{\partial\varphi}{\partial y}(x_0,y) = Q(x_0,y)$$

Therefore,

$$\frac{\partial\varphi}{\partial y} = Q(x,y).$$

The ODE then becomes

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$
$$\frac{\partial\varphi}{\partial x}\,dx + \frac{\partial\varphi}{\partial y}\,dy = 0.$$

On the left is how the differential of a two-dimensional function $\varphi = \varphi(x,y)$ is defined.

$$d\varphi = 0$$

Integrate both sides to obtain the ODE's solution.

$$\varphi(x,y) = \text{constant}$$