Exercise 7.3.2

Find the general solutions to the following ODEs. Write the solutions in forms that are entirely real (i.e., that contain no complex quantities).

$$y''' - 2y'' + y' - 2y = 0.$$

Solution

Because this is a linear homogeneous ODE and all the coefficients on the left side are constant, the solutions for it are of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt} \quad \rightarrow \quad y''' = r^3 e^{rt}$$

Substitute these formulas into the ODE.

$$r^3 e^{rt} - 2(r^2 e^{rt}) + re^{rt} - 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^3 - 2r^2 + r - 2 = 0$$

Solve for r.

$$(r-2)(r^2+1) = 0$$
$$(r-2)(r+i)(r-i) = 0$$
$$r = \{2, -i, i\}$$

Three solutions to the ODE are $y = e^{2t}$ and $y = e^{-it}$ and $y = e^{it}$. By the principle of superposition, the general solution is a linear combination of these three. Therefore,

$$\begin{aligned}
y(t) &= C_1 e^{2t} + C_2 e^{-it} + C_3 e^{it} \\
&= C_1 e^{2t} + C_2[\cos(-t) + i\sin(-t)] + C_3(\cos t + i \sin t) \\
&= C_1 e^{2t} + C_2(\cos t - i \sin t) + C_3(\cos t + i \sin t) \\
&= C_1 e^{2t} + C_2 \cos t - iC_2 \sin t + C_3 \cos t + iC_3 \sin t \\
&= C_1 e^{2t} + (C_2 + C_3)\cos t + (-iC_2 + iC_3)\sin t \\
&= C_1 e^{2t} + C_4 \cos t + C_5 \sin t.
\end{aligned}$$