Exercise 7.7.3

Find the general solutions to the following inhomogeneous ODEs:

$$y'' + 4y = e^x.$$

Solution

This is a linear ODE, so its general solution can be written as a sum of the complementary solution and the particular solution.

$$y(x) = y_c(x) + y_p(x)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 4y_c = 0$$

Because it's homogeneous and the coefficients on the left side are constant, the solution for y_c is of the form e^{rx}.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Substitute these formulas into the ODE.

$$r^2 e^{rx} + 4e^{rx} = 0$$

Divide both sides by e^{rx}.

$$r^2 + 4 = 0$$

$$r = \{-2i, 2i\}$$

Two solutions to the ODE are $y_c = e^{-2ix}$ and $y_c = e^{2ix}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned} y_c(x) &= C_1 e^{-2ix} + C_2 e^{2ix} \\ &= C_1[\cos(-2x) + i\sin(-2x)] + C_2[\cos(2x) + i\sin(2x)] \\ &= C_1(\cos 2x - i\sin 2x) + C_2(\cos 2x + i\sin 2x) \\ &= (C_1 + C_2)\cos 2x + (-iC_1 + iC_2)\sin 2x \\ &= C_3\cos 2x + C_4\sin 2x \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p'' + 4y_p = e^x. \tag{1}$$

Because the inhomogeneous term is an exponential function, y_p is expected to be an exponential function as well: $y_p(x) = Ae^x$. Substitute this formula into equation (1) to determine A.

$$(Ae^x)'' + 4(Ae^x) = e^x \quad \rightarrow \quad Ae^x + 4Ae^x = e^x \quad \rightarrow \quad 5A = 1 \quad \rightarrow \quad A = \frac{1}{5}$$

Therefore, the particular solution is $y_p(x) = (1/5)e^x$, and the general solution to the original ODE is

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= C_3\cos 2x + C_4\sin 2x + \frac{1}{5}e^x. \end{aligned}$$