Exercise 7.7.4

Find the general solutions to the following inhomogeneous ODEs:

$$y'' - 3y' + 2y = \sin x.$$

Solution

This is a linear ODE, so its general solution can be written as a sum of the complementary solution and the particular solution.

$$y(x) = y_c(x) + y_p(x)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 3y_c' + 2y_c = 0$$

Because it's homogeneous and the coefficients on the left side are constant, the solution for y_c is of the form e^{rx}.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Substitute these formulas into the ODE.

$$r^2 e^{rx} - 3(re^{rx}) + 2(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - 3r + 2 = 0$$

$$(r-1)(r-2) = 0$$

$$r = \{1, 2\}$$

Two solutions to the ODE are $y_c = e^x$ and $y_c = e^{2x}$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(x) = C_1 e^x + C_2 e^{2x}$$

On the other hand, the particular solution satisfies

$$y_p'' - 3y_p' + 2y_p = \sin x.$$

Because the inhomogeneous term is a sine function and there are even and odd derivatives on the left, y_p is expected to be a linear combination of sine and cosine: $y_p(x) = A\cos x + B\sin x$. Substitute this formula into the ODE to determine A and B.

$$(A\cos x + B\sin x)'' - 3(A\cos x + B\sin x)' + 2(A\cos x + B\sin x) = \sin x$$

$$(-A\cos x - B\sin x) - 3(-A\sin x + B\cos x) + 2(A\cos x + B\sin x) = \sin x$$

$$(-A - 3B + 2A)\cos x + (-B + 3A + 2B)\sin x = \sin x$$

Match the coefficients on both sides to obtain a system of equations for A and B.

$$-A - 3B + 2A = 0$$

$$-B + 3A + 2B = 1$$

Solving it yields

$$A = \frac{3}{10} \quad \text{and} \quad B = \frac{1}{10}.$$

Therefore, the particular solution is $y_p(x) = (3/10)\cos x + (1/10)\sin x$, and the general solution to the original ODE is

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= C_1 e^x + C_2 e^{2x} + \frac{3}{10}\cos x + \frac{1}{10}\sin x. \end{aligned}$$