Exercise 7.8.3

Solve the Bernoulli equation $y' + xy = xy^3$.

Solution

Divide both sides by y^3.

$$y^{-3}y' + xy^{-2} = x$$

Make the substitution $u = y^{-2}$. Then $u' = -2y^{-3}y'$ by the chain rule.

$$-\frac{1}{2}u' + xu = x$$

Multiply both sides by -2.

$$u' - 2xu = -2x$$

This is a first-order linear ODE that can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^x -2s\,ds\right) = e^{-x^2}$$

Proceed with the multiplication.

$$e^{-x^2}u' - 2xe^{-x^2}u = -2xe^{-x^2}$$

The left side can be written as $d/dx(Iu)$ by the product rule.

$$\frac{d}{dx}(e^{-x^2}u) = -2xe^{-x^2}$$

Integrate both sides with respect to x.

$$e^{-x^2}u = \int^x (-2se^{-s^2})\,ds + C$$

Let $v = -s^2$. Then $dv = -2s\,ds$.

$$\begin{aligned} e^{-x^2}u &= \int^{-x^2} e^v\,dv + C \\ &= e^{-x^2} + C \end{aligned}$$

Multiply both sides by e^{x^2}.

$$u(x) = 1 + Ce^{x^2}$$

Now that the ODE is solved, change back to y.

$$y^{-2} = 1 + Ce^{x^2}$$

Invert both sides.

$$y^2 = \frac{1}{1 + Ce^{x^2}}$$

Therefore, taking the square root of both sides,

$$y(x) = \pm\sqrt{\frac{1}{1 + Ce^{x^2}}}.$$