Exercise 7.2.17

Solve the ODE

$$(xy^2 - y)\,dx + x\,dy = 0.$$

Solution

This ODE is not exact at the moment because

$$\begin{gathered}\frac{\partial}{\partial y}(xy^2 - y) \neq \frac{\partial}{\partial x}(x) \\ 2xy - 1 \neq 1.\end{gathered}$$

In order to make it so, multiply both sides of the ODE by an integrating factor I.

$$(xy^2 - y)I\,dx + xI\,dy = 0.$$

Now that it is exact, we have

$$\begin{gathered}\frac{\partial}{\partial y}[(xy^2 - y)I] = \frac{\partial}{\partial x}(Ix) \\ (2xy - 1)I + (xy^2 - y)\frac{\partial I}{\partial y} = x\frac{\partial I}{\partial x} + I.\end{gathered}$$

To solve for a simple integrating factor, assume that it's only a function of y: $I = I(y)$.

$$(2xy - 1)I + (xy^2 - y)\frac{dI}{dy} = I$$

$$(2xy - 2)I + (xy^2 - y)\frac{dI}{dy} = 0$$

$$2(xy - 1)I + y(xy - 1)\frac{dI}{dy} = 0$$

Divide both sides by $xy - 1$.

$$2I + y\frac{dI}{dy} = 0$$

$$2 + y\frac{\frac{dI}{dy}}{I} = 0$$

$$\frac{\frac{dI}{dy}}{I} = -\frac{2}{y}$$

The left side can be written as the derivative of a logarithm.

$$\frac{d}{dy}\ln|I| = -\frac{2}{y}$$

Integrate both sides with respect to y.

$$\begin{aligned}\ln|I| &= -2\ln|y| + C_1 \\ &= \ln y^{-2} + C_1\end{aligned}$$

Exponentiate both sides.

$$\begin{aligned}|I| &= e^{\ln y^{-2}+C_1} \\ &= e^{\ln y^{-2}}e^{C_1} \\ &= y^{-2}e^{C_1}\end{aligned}$$

Remove the absolute value sign on the left by placing $\pm$ on the right.

$$I(y) = \pm e^{C_1} y^{-2}$$

Use a new constant C_2 for $\pm e^{C_1}$.

$$I(y) = C_2 y^{-2}$$

Any integrating factor will do, so choose $C_2 = 1$ for the simplest.

$$I(y) = y^{-2}$$

Now that the integrating factor is known, the original ODE can be solved.

$$(xy^2 - y)\,dx + x\,dy = 0$$

Multiply both sides by y^{-2}.

$$\left(x - \frac{1}{y}\right) dx + \frac{x}{y^2}\,dy = 0 \tag{1}$$

Since it is exact, there exists a potential function $\varphi = \varphi(x, y)$ that satisfies

$$\frac{\partial \varphi}{\partial x} = x - \frac{1}{y} \tag{2}$$

$$\frac{\partial \varphi}{\partial y} = \frac{x}{y^2}. \tag{3}$$

As a result, equation (1) can be written as

$$\frac{\partial \varphi}{\partial x}\,dx + \frac{\partial \varphi}{\partial y}\,dy = 0.$$

The left side is how the differential of φ is defined.

$$d\varphi = 0$$

Integrate both sides.

$$\varphi(x, y) = C_3$$

The general solution to the ODE is found then by solving equations (2) and (3) for φ. Integrate both sides of equation (2) partially with respect to x to get φ.

$$\varphi(x, y) = \frac{1}{2}x^2 - \frac{x}{y} + f(y)$$

Differentiate both sides with respect to y.

$$\frac{\partial \varphi}{\partial y} = \frac{x}{y^2} + f'(y)$$

Comparing this formula for $\partial\varphi/\partial y$ to equation (3), we see that

$$f'(y) = 0.$$

Integrate both sides with respect to y.

$$f(y) = C_4$$

Consequently, the potential function is

$$\varphi(x, y) = \frac{1}{2}x^2 - \frac{x}{y} + C_4,$$

and the general solution to the ODE is

$$\frac{1}{2}x^2 - \frac{x}{y} = C_5,$$

where C_5 is a new constant used for $C_3 - C_4$. Solve this equation for y.

$$\frac{x}{y} = \frac{x^2}{2} - C_5$$

Invert both sides.

$$\begin{aligned} \frac{y}{x} &= \frac{1}{\frac{x^2}{2} - C_5} \\ &= \frac{2}{x^2 - 2C_5} \end{aligned}$$

Therefore, multiplying both sides by x and using a new constant A for $-2C_5$,

$$y(x) = \frac{2x}{x^2 + A}.$$