Exercise 7.2.3

The decay of a population by catastrophic two-body collisions is described by

$$\frac{dN}{dt} = -kN^2.$$

This is a first-order, **nonlinear** differential equation. Derive the solution

$$N(t) = N_0 \left(1 + \frac{t}{\tau_0}\right)^{-1},$$

where $\tau_0 = (kN_0)^{-1}$. This implies an infinite population at $t = -\tau_0$.

Solution

Divide both sides of the ODE by N^2.

$$\frac{dN/dt}{N^2} = -k$$

The left side can be written as a derivative by the chain rule.

$$-\frac{d}{dt}\left(\frac{1}{N}\right) = -k$$

Multiply both sides by -1.

$$\frac{d}{dt}\left(\frac{1}{N}\right) = k$$

Integrate both sides with respect to t.

$$\frac{1}{N} = kt + C$$

Use the initial condition $N(0) = N_0$ to determine C.

$$\frac{1}{N_0} = C$$

Then the previous equation becomes

$$\frac{1}{N} = kt + \frac{1}{N_0}.$$

Invert both sides to get N.

$$N(t) = \frac{1}{kt + \frac{1}{N_0}}$$

Multiply the numerator and denominator by N_0.

$$\begin{aligned} N(t) &= \frac{N_0}{kN_0 t + 1} \\ &= N_0 \frac{1}{1 + \frac{t}{(kN_0)^{-1}}} \\ &= N_0 \left(1 + \frac{t}{(kN_0)^{-1}}\right)^{-1} \end{aligned}$$

Therefore,

$$N(t) = N_0 \left(1 + \frac{t}{\tau_0}\right)^{-1},$$

where $\tau_0 = (kN_0)^{-1}$.