Exercise 7.2.4

The rate of a particular chemical reaction $A + B \to C$ is proportional to the concentrations of the reactants A and B:

$$\frac{dC(t)}{dt} = \alpha[A(0) - C(t)][B(0) - C(t)].$$

(a) Find $C(t)$ for $A(0) \neq B(0)$.

(b) Find $C(t)$ for $A(0) = B(0)$.

The initial condition is that $C(0) = 0$.

Solution

Part (a)

Solve the ODE by separating variables. Divide both sides by $[A(0) - C(t)][B(0) - C(t)]$.

$$\frac{dC}{[A(0) - C][B(0) - C]} = \alpha \, dt$$

Integrate both sides.

$$\int^C \frac{dx}{[A(0) - x][B(0) - x]} = \int^t \alpha \, ds$$

Use partial fraction decomposition on the left and evaluate the integral on the right.

$$\int^C \left[\frac{1/[B(0) - A(0)]}{A(0) - x} + \frac{1/[A(0) - B(0)]}{B(0) - x}\right] dx = \alpha t + D$$

$$\frac{1}{B(0) - A(0)} \int^C \frac{dx}{A(0) - x} + \frac{1}{A(0) - B(0)} \int^C \frac{dx}{B(0) - x} = \alpha t + D$$

$$-\frac{1}{B(0) - A(0)} \int^C \frac{dx}{x - A(0)} - \frac{1}{A(0) - B(0)} \int^C \frac{dx}{x - B(0)} = \alpha t + D$$

Evaluate the integrals.

$$\frac{1}{A(0) - B(0)} \ln|x - A(0)|\Big|^C - \frac{1}{A(0) - B(0)} \ln|x - B(0)|\Big|^C = \alpha t + D$$

$$\frac{1}{A(0) - B(0)} \ln|C(t) - A(0)| - \frac{1}{A(0) - B(0)} \ln|C(t) - B(0)| = \alpha t + D$$

$$\frac{1}{A(0) - B(0)} [\ln|C(t) - A(0)| - \ln|C(t) - B(0)|] = \alpha t + D$$

$$\frac{1}{A(0) - B(0)} \ln\left|\frac{C(t) - A(0)}{C(t) - B(0)}\right| = \alpha t + D$$

Multiply both sides by $A(0) - B(0)$, using E for $D[A(0) - B(0)]$.

$$\ln\left|\frac{C(t) - A(0)}{C(t) - B(0)}\right| = \alpha[A(0) - B(0)]t + E$$

Exponentiate both sides.

$$\begin{aligned}\left|\frac{C(t)-A(0)}{C(t)-B(0)}\right| &= e^{\alpha[A(0)-B(0)]t+E} \\ &= e^{\alpha[A(0)-B(0)]t}e^{E}\end{aligned}$$

Remove the absolute value sign on the left by placing $\pm$ on the right side.

$$\frac{C(t)-A(0)}{C(t)-B(0)} = \pm e^{E}e^{\alpha[A(0)-B(0)]t}$$

Use a new constant F for $\pm e^{E}$.

$$\frac{C(t)-A(0)}{C(t)-B(0)} = Fe^{\alpha[A(0)-B(0)]t}$$

Solve this equation for $C(t)$.

$$\begin{aligned}C(t)-A(0) &= FC(t)e^{\alpha[A(0)-B(0)]t} - FB(0)e^{\alpha[A(0)-B(0)]t} \\ C(t)-FC(t)e^{\alpha[A(0)-B(0)]t} &= A(0) - FB(0)e^{\alpha[A(0)-B(0)]t} \\ C(t)\left\{1-Fe^{\alpha[A(0)-B(0)]t}\right\} &= A(0) - FB(0)e^{\alpha[A(0)-B(0)]t} \\ C(t) &= \frac{A(0)-FB(0)e^{\alpha[A(0)-B(0)]t}}{1-Fe^{\alpha[A(0)-B(0)]t}}\end{aligned}$$

Now apply the initial condition $C(0) = 0$ to determine F.

$$C(0) = \frac{A(0)-FB(0)}{1-F} = 0 \quad \rightarrow \quad A(0)-FB(0) = 0 \quad \rightarrow \quad F = \frac{A(0)}{B(0)}$$

Therefore, if $A(0) \neq B(0)$,

$$\begin{aligned}C(t) &= \frac{A(0) - \left[\frac{A(0)}{B(0)}\right]B(0)e^{\alpha[A(0)-B(0)]t}}{1-\left[\frac{A(0)}{B(0)}\right]e^{\alpha[A(0)-B(0)]t}} \\ &= \frac{A(0)-A(0)e^{\alpha[A(0)-B(0)]t}}{1-\frac{A(0)}{B(0)}e^{\alpha[A(0)-B(0)]t}} \\ &= A(0)\frac{1-e^{\alpha[A(0)-B(0)]t}}{1-\frac{A(0)}{B(0)}e^{\alpha[A(0)-B(0)]t}}\cdot\frac{B(0)}{B(0)} \\ &= A(0)B(0)\frac{1-e^{\alpha[A(0)-B(0)]t}}{B(0)-A(0)e^{\alpha[A(0)-B(0)]t}}.\end{aligned}$$

Part (b)

Set $A(0) = B(0)$ in the ODE.

$$\begin{aligned}\frac{dC(t)}{dt} &= \alpha[A(0) - C(t)][B(0) - C(t)] \\ &= \alpha[B(0) - C(t)][B(0) - C(t)] \\ &= \alpha[B(0) - C(t)]^2 \\ &= \alpha[C(t) - B(0)]^2\end{aligned}$$

Solve it by separating variables. Divide both sides by $[C(t) - B(0)]^2$.

$$\frac{dC}{[C(t) - B(0)]^2} = \alpha\, dt$$

Integrate both sides.

$$\int^C \frac{dx}{[x - B(0)]^2} = \int^t \alpha\, ds$$

Let $u = x - B(0)$ in the integral on the left side. Then $du = dx$.

$$\int^{C-B(0)} \frac{du}{u^2} = \int^t \alpha\, ds$$

Evaluate the integrals.

$$-\frac{1}{u}\bigg|^{C-B(0)} = \alpha t + G$$

$$-\frac{1}{C(t) - B(0)} = \alpha t + G$$

Apply the initial condition $C(0) = 0$ now to determine G.

$$-\frac{1}{C(0) - B(0)} = \alpha(0) + G \quad \rightarrow \quad -\frac{1}{-B(0)} = G \quad \rightarrow \quad G = \frac{1}{B(0)}$$

Then the previous equation becomes

$$-\frac{1}{C(t) - B(0)} = \alpha t + \frac{1}{B(0)}.$$

Solve for $C(t)$.

$$-[C(t) - B(0)] = \frac{1}{\alpha t + \frac{1}{B(0)}}$$

$$C(t) - B(0) = -\frac{1}{\alpha t + \frac{1}{B(0)}}$$

$$\begin{aligned}C(t) &= B(0) - \frac{1}{\alpha t + \frac{1}{B(0)}} \cdot \frac{B(0)}{B(0)} \\ &= B(0) - \frac{B(0)}{\alpha B(0) t + 1}\end{aligned}$$

Therefore, if $A(0) = B(0)$,

$$C(t) = B(0)\left[1 - \frac{1}{\alpha B(0) t + 1}\right].$$