Exercise 7.3.3

Find the general solutions to the following ODEs. Write the solutions in forms that are entirely real (i.e., that contain no complex quantities).

$$y''' - 3y' + 2y = 0.$$

Solution

Because this is a linear homogeneous ODE and all the coefficients on the left side are constant, the solutions for it are of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2 e^{rt} \quad \to \quad y''' = r^3 e^{rt}$$

Substitute these formulas into the ODE.

$$r^3 e^{rt} - 3(re^{rt}) + 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^3 - 3r + 2 = 0$$

Solve for r.

$$(r+2)(r-1)^2 = 0$$

$$r = \{-2, 1\}$$

Two solutions to the ODE are $y = e^{-2t}$ and $y = e^t$. But since the order of this ODE is 3, there must be a third solution. Use the method of reduction of order to find the general solution: Plug in $y(t) = c(t)e^t$ to the ODE and solve the resulting equation for $c(t)$.

$$[c(t)e^t]''' - 3[c(t)e^t]' + 2[c(t)e^t] = 0$$

$$[c'(t)e^t + c(t)e^t]'' - 3[c'(t)e^t + c(t)e^t] + 2c(t)e^t = 0$$

$$[c''(t)e^t + 2c'(t)e^t + c(t)e^t]' - 3c'(t)e^t - c(t)e^t = 0$$

$$[c'''(t)e^t + 3c''(t)e^t + \cancel{3c'(t)e^t} + \cancel{c(t)e^t}] - \cancel{3c'(t)e^t} - \cancel{c(t)e^t} = 0$$

$$c'''(t)e^t + 3c''(t)e^t = 0$$

Divide both sides by $c''(t)e^t$ and bring 3 to the right side.

$$\frac{c'''(t)}{c''(t)} = -3$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{dt} \ln |c''(t)| = -3$$

Integrate both sides with respect to t.

$$\ln |c''(t)| = -3t + C_1$$

Exponentiate both sides.

$$\begin{aligned}|c''(t)| &= e^{-3t+C_1} \\ &= e^{-3t}e^{C_1}\end{aligned}$$

Remove the absolute value sign on the left by placing $\pm$ on the right side.

$$c''(t) = \pm e^{C_1}e^{-3t}$$

Use a new constant C_2 for $\pm e^{C_1}$.

$$c''(t) = C_2 e^{-3t}$$

Integrate both sides with respect to t.

$$c'(t) = -\frac{C_2}{3}e^{-3t} + C_3$$

Integrate both sides with respect to t once more.

$$c(t) = \frac{C_2}{9}e^{-3t} + C_3 t + C_4$$

Since $y(t) = c(t)e^t$,

$$y(t) = \frac{C_2}{9}e^{-2t} + C_3 te^t + C_4 e^t.$$

Therefore, using a new constant C_5 for $C_2/9$,

$$y(t) = C_5 e^{-2t} + (C_3 t + C_4)e^t.$$