Exercise 7.4.5

Show that the substitution

$$x \to \frac{1-x}{2}, \quad a = -l, \quad b = l+1, \quad c = 1$$

converts the hypergeometric equation into Legendre's equation.

[TYPO: The hypergeometric equation listed in the text in Table 7.1 on page 345 is incorrect and will not lead to Legendre's equation.]

Solution

The hypergeometric equation is a second-order linear homogeneous ODE and has a minus sign in front of c.

$$x(x-1)y'' + [(1+a+b)x - c]y' + aby = 0$$

In order to change this into the Legendre equation, make the substitution,

$$x = \frac{1-z}{2}.$$

It becomes

$$\frac{1-z}{2}\left(\frac{1-z}{2} - 1\right)y'' + \left[(1+a+b)\frac{1-z}{2} - c\right]y' + aby = 0.$$

Use the chain rule to find what the derivatives of y are in terms of this new variable ($z = 1-2x$).

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}(-2) = -2\frac{dy}{dz}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dz}{dx}\frac{d}{dz}\left(-2\frac{dy}{dz}\right) = -2\left(-2\frac{d^2y}{dz^2}\right) = 4\frac{d^2y}{dz^2}$$

As a result, the ODE in terms of z is

$$\frac{1-z}{2}\left(\frac{1-z}{2} - 1\right)\left(4\frac{d^2y}{dz^2}\right) + \left[(1+a+b)\frac{1-z}{2} - c\right]\left(-2\frac{dy}{dz}\right) + aby = 0,$$

or after simplifying,

$$-(1-z^2)\frac{d^2y}{dz^2} + \left[(1+a+b)\frac{1-z}{2} - c\right]\left(-2\frac{dy}{dz}\right) + aby = 0.$$

Now set $a = -l$, $b = l+1$, and $c = 1$.

$$-(1-z^2)\frac{d^2y}{dz^2} + \left[(2)\frac{1-z}{2} - 1\right]\left(-2\frac{dy}{dz}\right) - l(l+1)y = 0$$

$$-(1-z^2)\frac{d^2y}{dz^2} + (-z)\left(-2\frac{dy}{dz}\right) - l(l+1)y = 0$$

$$-(1-z^2)\frac{d^2y}{dz^2} + 2z\frac{dy}{dz} - l(l+1)y = 0$$

Therefore, multiplying both sides by -1, the Legendre equation is obtained.

$$(1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + l(l+1)y = 0$$