Exercise 7.7.1

If our linear, second-order ODE is inhomogeneous, that is, of the form of Eq. (7.94), the **most general solution is**

$$y(x) = y_1(x) + y_2(x) + y_p(x),$$

where y_1 and y_2 are independent solutions of the homogeneous equation. Show that

$$y_p(x) = y_2(x) \int^x \frac{y_1(s)F(s)\,ds}{W\{y_1(s), y_2(s)\}} - y_1(x) \int^x \frac{y_2(s)F(s)\,ds}{W\{y_1(s), y_2(s)\}},$$

with $W\{y_1(x), y_2(x)\}$ the Wronskian of $y_1(s)$ and $y_2(s)$.

[TYPO: These should be $y_1(x)$ and $y_2(x)$.]

Solution

Eq. (7.94) is the general linear second-order inhomogeneous ODE.

$$y'' + P(x)y' + Q(x)y = F(x) \tag{7.94}$$

Because it's linear, the general solution can be written as a sum of the complementary solution and the particular solution.

$$y(x) = y_c(x) + y_p(x)$$

The complementary solution satisfies the associated homogeneous ODE.

$$y_c'' + P(x)y_c' + Q(x)y_c = 0$$

Suppose that two linearly independent solutions for $y_c(x)$ are $y_1(x)$ and $y_2(x)$. Then, by the principle of superposition, the general solution is $y_c(x) = C_1y_1(x) + C_2y_2(x)$. On the other hand, the particular solution satisfies

$$y_p'' + P(x)y_p' + Q(x)y_p = F(x).$$

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in y_c to vary: $y_p(x) = C_1(x)y_1(x) + C_2(x)y_2(x)$. Substitute this formula into the ODE and simplify the result.

$$[C_1(x)y_1(x) + C_2(x)y_2(x)]'' + P(x)[C_1(x)y_1(x) + C_2(x)y_2(x)]' + Q(x)[C_1(x)y_1(x) + C_2(x)y_2(x)] = F(x)$$

$$[C_1'(x)y_1(x) + C_1(x)y_1'(x) + C_2'(x)y_2(x) + C_2(x)y_2'(x)]' + P(x)[C_1'(x)y_1(x) + C_1(x)y_1'(x) + C_2'(x)y_2(x) + C_2(x)y_2'(x)] + Q(x)[C_1(x)y_1(x) + C_2(x)y_2(x)] = F(x)$$

$$[C_1''(x)y_1(x) + 2C_1'(x)y_1'(x) + C_1(x)y_1''(x) + C_2''(x)y_2(x) + 2C_2'(x)y_2'(x) + C_2(x)y_2''(x)] + P(x)[C_1'(x)y_1(x) + C_1(x)y_1'(x) + C_2'(x)y_2(x) + C_2(x)y_2'(x)] + Q(x)[C_1(x)y_1(x) + C_2(x)y_2(x)] = F(x)$$

Factor the terms with $C_1(x)$ and $C_2(x)$. Because y_1 and y_2 satisfy the associated homogeneous equation, they all add to zero.

$$\begin{aligned}[C_1''(x)y_1(x) + 2C_1'(x)y_1'(x) + C_2''(x)y_2(x) + 2C_2'(x)y_2'(x)] + P(x)[C_1'(x)y_1(x) + C_2'(x)y_2(x)] \\ C_1(x)\underbrace{[y_1''(x) + P(x)y_1'(x) + Q(x)y_1(x)]}_{=0} + C_2(x)\underbrace{[y_2''(x) + P(x)y_2'(x) + Q(x)y_2(x)]}_{=0} = F(x)\end{aligned}$$

If we set

$$C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0, \tag{1}$$

then the previous equation reduces to

$$\begin{aligned}C_1''(x)y_1(x) + 2C_1'(x)y_1'(x) + C_2''(x)y_2(x) + 2C_2'(x)y_2'(x) &= F(x) \\ [C_1'(x)y_1(x)]' + C_1'(x)y_1'(x) + [C_2'(x)y_2(x)]' + C_2'(x)y_2'(x) &= F(x). \end{aligned} \tag{2}$$

We now have a system of two equations for the two unknowns, $C_1(x)$ and $C_2(x)$. Solve equation (1) for $C_2'(x)$

$$C_2'(x) = -\frac{y_1(x)}{y_2(x)}C_1'(x) \tag{3}$$

and then plug this result into equation (2) to get one solely for $C_1'(x)$.

$$C_1''(x)y_1(x) + 2C_1'(x)y_1'(x) + \left[-\frac{y_1(x)}{y_2(x)}C_1'(x)\right]' y_2(x) + 2\left[-\frac{y_1(x)}{y_2(x)}C_1'(x)\right] y_2'(x) = F(x)$$

$$\begin{aligned}\cancel{C_1''(x)y_1(x)} + 2C_1'(x)y_1'(x) - \left[\cancel{\frac{y_1(x)}{y_2(x)}C_1''(x)} + \frac{y_1'(x)y_2(x) - y_2'(x)y_1(x)}{[y_2(x)]^2}C_1'(x)\right] y_2(x) \\ + 2\left[-\frac{y_1(x)}{y_2(x)}C_1'(x)\right] y_2'(x) = F(x)\end{aligned}$$

$$\begin{aligned}2C_1'(x)y_1'(x) - \frac{y_1'(x)y_2(x) - y_2'(x)y_1(x)}{y_2(x)}C_1'(x) - \frac{2y_2'(x)y_1(x)}{y_2(x)}C_1'(x) &= F(x) \\ C_1'(x)\left[2y_1'(x) - \frac{y_1'(x)y_2(x) - y_2'(x)y_1(x)}{y_2(x)} - \frac{2y_2'(x)y_1(x)}{y_2(x)}\right] &= F(x) \\ C_1'(x)\left[\frac{y_1'(x)y_2(x) - y_2'(x)y_1(x)}{y_2(x)}\right] &= F(x)\end{aligned}$$

Solve for $C_1'(x)$.

$$C_1'(x) = -\frac{y_2(x)}{y_2'(x)y_1(x) - y_1'(x)y_2(x)}F(x)$$

Integrate both sides with respect to x, setting the integration constant to zero.

$$C_1(x) = \int^x -\frac{y_2(s)}{y_2'(s)y_1(s) - y_1'(s)y_2(s)}F(s)\,ds$$

Now substitute the previous formula for $C_1'(x)$ into equation (3) to get $C_2(x)$.

$$\begin{aligned} C_2'(x) &= -\frac{y_1(x)}{y_2(x)}C_1'(x) \\ &= -\frac{y_1(x)}{y_2(x)}\left[-\frac{y_2(x)}{y_2'(x)y_1(x)-y_1'(x)y_2(x)}F(x)\right] \\ &= \frac{y_1(x)}{y_2'(x)y_1(x)-y_1'(x)y_2(x)}F(x) \end{aligned}$$

Integrate both sides with respect to x, setting the integration constant to zero.

$$C_2(x) = \int^x \frac{y_1(s)}{y_2'(s)y_1(s)-y_1'(s)y_2(s)}F(s)\,ds$$

Now that $C_1(x)$ and $C_2(x)$ are known, the particular solution is as well.

$$\begin{aligned} y_p(x) &= C_1(x)y_1(x) + C_2(x)y_2(x) \\ &= y_1(x)\int^x -\frac{y_2(s)}{y_2'(s)y_1(s)-y_1'(s)y_2(s)}F(s)\,ds + y_2(x)\int^x \frac{y_1(s)}{y_2'(s)y_1(s)-y_1'(s)y_2(s)}F(s)\,ds \\ &= -\int^x \frac{y_1(x)y_2(s)}{y_2'(s)y_1(s)-y_1'(s)y_2(s)}F(s)\,ds + \int^x \frac{y_1(s)y_2(x)}{y_2'(s)y_1(s)-y_1'(s)y_2(s)}F(s)\,ds \\ &= \int^x \left[-\frac{y_1(x)y_2(s)}{y_2'(s)y_1(s)-y_1'(s)y_2(s)}F(s) + \frac{y_1(s)y_2(x)}{y_2'(s)y_1(s)-y_1'(s)y_2(s)}F(s)\right]ds \\ &= \int^x \frac{y_1(s)y_2(x)-y_1(x)y_2(s)}{y_2'(s)y_1(s)-y_1'(s)y_2(s)}F(s)\,ds \\ &= \int^x \frac{y_1(s)y_2(x)-y_1(x)y_2(s)}{W\{y_1(s),y_2(s)\}}F(s)\,ds \end{aligned}$$

Note that the Wronskian of two functions can be written as a determinant.

$$W\{y_1,y_2\} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2.$$

Therefore, the general solution to the inhomogeneous ODE is

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= C_1y_1(x) + C_2y_2(x) + \int^x \frac{y_1(s)y_2(x)-y_1(x)y_2(s)}{W\{y_1(s),y_2(s)\}}F(s)\,ds. \end{aligned}$$