Exercise 9.2.6

Find the general solution to the PDE

$$x\frac{\partial \psi}{\partial x} - y\frac{\partial \psi}{\partial y} = 0.$$

Hint. The solution to Exercise 9.2.5 may provide a suggestion as to how to proceed.

Solution

Divide both sides by x.

$$\frac{\partial \psi}{\partial x} - \frac{y}{x}\frac{\partial \psi}{\partial y} = 0. \tag{1}$$

Since ψ is a function of two variables $\psi = \psi(x, y)$, its differential is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the relationship between the total derivative of ψ and the partial derivatives of ψ.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{dy}{dx}\frac{\partial \psi}{\partial y}$$

In light of this, equation (1) reduces to the ODE,

$$\frac{d\psi}{dx} = 0, \tag{2}$$

along the characteristic curves in the xy-plane that satisfy

$$\frac{dy}{dx} = -\frac{y}{x}, \qquad y(1, \xi) = \xi, \tag{3}$$

where ξ is a characteristic coordinate. Solve for $y(x, \xi)$ by separating variables.

$$\frac{dy}{y} = -\frac{dx}{x}$$

Integrate both sides.

$$\ln|y| = -\ln|x| + \eta$$

Exponentiate both sides.

$$\begin{aligned} |y| &= e^{-\ln|x| + \eta} \\ &= e^{\eta} e^{\ln|x|^{-1}} \\ &= e^{\eta}|x|^{-1} \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign around y.

$$y(x, \eta) = \frac{\pm e^{\eta}}{|x|}$$

Let ξ be $\pm e^{\eta}$.

$$y(x, \xi) = \frac{\xi}{|x|} \quad \rightarrow \quad \xi = |x|y$$

Integrate both sides of equation (2) with respect to x.

$$\psi(x, \xi) = f(\xi)$$

f is an arbitrary function of the characteristic coordinate. Now eliminate ξ in favor of x and y.

$$\psi(x, y) = f(|x|y)$$

The absolute value sign can be dropped safely because f is arbitrary. Therefore,

$$\psi(x, y) = f(xy).$$