Exercise 9.3.1

Show that by making a change of variables to $\xi = c^{1/2}x - c^{-1/2}by$, $\eta = c^{-1/2}y$, the operator $\mathcal{L}$ of Eq. (9.18) can be brought to the form

$$\mathcal{L} = (ac - b^2)\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}.$$

Solution

Consider the following sum of second derivatives of a function $\varphi(x, y)$.

$$\begin{aligned} a\frac{\partial^2\varphi}{\partial x^2} + 2b\frac{\partial^2\varphi}{\partial x\partial y} + c\frac{\partial^2\varphi}{\partial y^2} &= \left(a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x\partial y} + c\frac{\partial^2}{\partial y^2}\right)\varphi \\ &= \left(\frac{b + \sqrt{b^2 - ac}}{c^{1/2}}\frac{\partial}{\partial x} + c^{1/2}\frac{\partial}{\partial y}\right)\left(\frac{b - \sqrt{b^2 - ac}}{c^{1/2}}\frac{\partial}{\partial x} + c^{1/2}\frac{\partial}{\partial y}\right)\varphi \\ &= \mathcal{L}\varphi \end{aligned}$$

Eq. (9.18) in the text is

$$\mathcal{L} = \left(\frac{b + \sqrt{b^2 - ac}}{c^{1/2}}\frac{\partial}{\partial x} + c^{1/2}\frac{\partial}{\partial y}\right)\left(\frac{b - \sqrt{b^2 - ac}}{c^{1/2}}\frac{\partial}{\partial x} + c^{1/2}\frac{\partial}{\partial y}\right). \tag{9.18}$$

As instructed, make the change of variables,

$$\xi = c^{1/2}x - c^{-1/2}by \qquad \eta = c^{-1/2}y.$$

The aim now is to find $\partial/\partial x$ and $\partial/\partial y$ in terms of these new variables. Use the chain rule.

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \xi}(c^{1/2}) + \frac{\partial}{\partial \eta}(0) = c^{1/2}\frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial \xi}(-c^{-1/2}b) + \frac{\partial}{\partial \eta}(c^{-1/2}) = c^{-1/2}\left(\frac{\partial}{\partial \eta} - b\frac{\partial}{\partial \xi}\right) \end{aligned}$$

Substitute these results into the operator $\mathcal{L}$.

$$\begin{aligned} \mathcal{L} &= \left[\frac{b + \sqrt{b^2 - ac}}{c^{1/2}}c^{1/2}\frac{\partial}{\partial \xi} + c^{1/2}c^{-1/2}\left(\frac{\partial}{\partial \eta} - b\frac{\partial}{\partial \xi}\right)\right]\left[\frac{b - \sqrt{b^2 - ac}}{c^{1/2}}c^{1/2}\frac{\partial}{\partial \xi} + c^{1/2}c^{-1/2}\left(\frac{\partial}{\partial \eta} - b\frac{\partial}{\partial \xi}\right)\right] \\ &= \left[(b + \sqrt{b^2 - ac})\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - b\frac{\partial}{\partial \xi}\right]\left[(b - \sqrt{b^2 - ac})\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - b\frac{\partial}{\partial \xi}\right] \\ &= \left(b\frac{\partial}{\partial \xi} + \sqrt{b^2 - ac}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - b\frac{\partial}{\partial \xi}\right)\left(b\frac{\partial}{\partial \xi} - \sqrt{b^2 - ac}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - b\frac{\partial}{\partial \xi}\right) \\ &= \left(\sqrt{b^2 - ac}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(-\sqrt{b^2 - ac}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) \\ &= -(b^2 - ac)\frac{\partial^2}{\partial \xi^2} + \sqrt{b^2 - ac}\frac{\partial}{\partial \xi\partial \eta} - \sqrt{b^2 - ac}\frac{\partial}{\partial \eta\partial \xi} + \frac{\partial^2}{\partial \eta^2} \end{aligned}$$

Therefore,

$$\mathcal{L} = (ac - b^2)\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}.$$