Exercise 9.4.2

Show that the Helmholtz equation,

$$\nabla^2\psi + k^2\psi = 0,$$

is still separable in circular cylindrical coordinates if k^2 is generalized to $k^2 + f(\rho) + (1/\rho^2)g(\varphi) + h(z)$.

Solution

Replace k^2 with $k^2 + f(\rho) + (1/\rho^2)g(\varphi) + h(z)$ in the Helmholtz equation.

$$\nabla^2\psi + \left[k^2 + f(\rho) + \frac{1}{\rho^2}g(\varphi) + h(z)\right]\psi = 0$$

Expand the Laplacian operator in circular cylindrical coordinates (ρ, φ, z).

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2} + \left[k^2 + f(\rho) + \frac{1}{\rho^2}g(\varphi) + h(z)\right]\psi = 0$$

Assume a product solution of the form $\psi(\rho, \varphi, z) = P(\rho)\Phi(\varphi)Z(z)$ and substitute it into the PDE.

$$\begin{aligned}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}[P(\rho)\Phi(\varphi)Z(z)]\right] + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}[P(\rho)\Phi(\varphi)Z(z)] + \frac{\partial^2}{\partial z^2}[P(\rho)\Phi(\varphi)Z(z)]& \\ + \left[k^2 + f(\rho) + \frac{1}{\rho^2}g(\varphi) + h(z)\right]P(\rho)\Phi(\varphi)Z(z) &= 0\end{aligned}$$

$$\begin{aligned}\frac{\Phi(\varphi)Z(z)}{\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \frac{P(\rho)Z(z)}{\rho^2}\Phi''(\varphi) + P(\rho)\Phi(\varphi)Z''(z)& \\ + \left[k^2 + f(\rho) + \frac{1}{\rho^2}g(\varphi) + h(z)\right]P(\rho)\Phi(\varphi)Z(z) &= 0\end{aligned}$$

Divide both sides by $P(\rho)\Phi(\varphi)Z(z)$.

$$\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z} + k^2 + f(\rho) + \frac{1}{\rho^2}g(\varphi) + h(z) = 0$$

$$\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + f(\rho) + \frac{1}{\rho^2}\left[\frac{\Phi''}{\Phi} + g(\varphi)\right] + \frac{Z''}{Z} + h(z) + k^2 = 0$$

Bring the first three terms over to the right side.

$$\underbrace{\frac{Z''}{Z} + h(z) + k^2}_{\text{function of } z} = \underbrace{-\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) - f(\rho) - \frac{1}{\rho^2}\left[\frac{\Phi''}{\Phi} + g(\varphi)\right]}_{\text{function of } \rho \text{ and } \varphi}$$

The only way a function of z can be equal to a function of ρ and φ is if both are equal to a constant λ.

$$\frac{Z''}{Z} + h(z) + k^2 = -\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) - f(\rho) - \frac{1}{\rho^2}\left[\frac{\Phi''}{\Phi} + g(\varphi)\right] = \lambda$$

The second equation is

$$-\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) - f(\rho) - \frac{1}{\rho^2}\left[\frac{\Phi''}{\Phi} + g(\varphi)\right] = \lambda.$$

Bring the third term to the right side and bring λ to the left side.

$$-\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) - f(\rho) - \lambda = \frac{1}{\rho^2}\left[\frac{\Phi''}{\Phi} + g(\varphi)\right]$$

Multiply both sides by $-\rho^2$.

$$\underbrace{\frac{\rho}{P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \rho^2[f(\rho) + \lambda]}_{\text{function of } \rho} = \underbrace{-\left[\frac{\Phi''}{\Phi} + g(\varphi)\right]}_{\text{function of } \varphi}$$

The only way a function of ρ can be equal to a function of φ is if both are equal to another constant μ.

$$\frac{\rho}{P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \rho^2[f(\rho) + \lambda] = -\left[\frac{\Phi''}{\Phi} + g(\varphi)\right] = \mu$$

In summary, as a result of assuming a product solution, the Helmholtz equation in circular cylindrical coordinates has reduced to three ODEs—one in ρ, one in φ, and one in z.

$$\left.\begin{aligned} \frac{\rho}{P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \rho^2[f(\rho) + \lambda] &= \mu \\ -\left[\frac{\Phi''}{\Phi} + g(\varphi)\right] &= \mu \\ \frac{Z''}{Z} + h(z) + k^2 &= \lambda \end{aligned}\right\}$$

Therefore, the Helmholtz equation is still separable in circular cylindrical coordinates if k^2 is generalized to $k^2 + f(\rho) + (1/\rho^2)g(\varphi) + h(z)$.