Exercise 9.4.4

Verify that

$$\nabla^2\psi(r,\theta,\varphi) + \left[k^2 + f(r) + \frac{1}{r^2}g(\theta) + \frac{1}{r^2\sin^2\theta}h(\varphi)\right]\psi(r,\theta,\varphi) = 0$$

is separable (in spherical polar coordinates). The functions f, g, and h are functions only of the variables indicated; k^2 is a constant.

Solution

Expand the Laplacian operator in spherical polar coordinates (r,θ,φ). Here θ is the angle from the polar axis.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} + \left[k^2 + f(r) + \frac{1}{r^2}g(\theta) + \frac{1}{r^2\sin^2\theta}h(\varphi)\right]\psi = 0$$

Assume a product solution of the form $\psi(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi)$ and substitute it into the PDE.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}[R(r)\Theta(\theta)\Phi(\varphi)]\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}[R(r)\Theta(\theta)\Phi(\varphi)]\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}[R(r)\Theta(\theta)\Phi(\varphi)] + \left[k^2 + f(r) + \frac{1}{r^2}g(\theta) + \frac{1}{r^2\sin^2\theta}h(\varphi)\right]R(r)\Theta(\theta)\Phi(\varphi) = 0$$

$$\frac{\Theta(\theta)\Phi(\varphi)}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R(r)\Phi(\varphi)}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{R(r)\Theta(\theta)}{r^2\sin^2\theta}\Phi''(\varphi) + \left[k^2 + f(r) + \frac{1}{r^2}g(\theta) + \frac{1}{r^2\sin^2\theta}h(\varphi)\right]R(r)\Theta(\theta)\Phi(\varphi) = 0$$

Divide both sides by $R(r)\Theta(\theta)\Phi(\varphi)$.

$$\frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{r^2\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi} + k^2 + f(r) + \frac{1}{r^2}g(\theta) + \frac{1}{r^2\sin^2\theta}h(\varphi) = 0$$

$$\frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{r^2\sin^2\theta}\left[\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{\Phi''}{\Phi} + g(\theta)\sin^2\theta + h(\varphi)\right] + f(r) + k^2 = 0$$

Multiply both sides by r^2 and bring the second term to the right side.

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + f(r) + k^2}_{\text{function of } r} = \underbrace{-\frac{1}{\sin^2\theta}\left[\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{\Phi''}{\Phi} + g(\theta)\sin^2\theta + h(\varphi)\right]}_{\text{function of } \theta \text{ and } \varphi}$$

The only way a function of r can be equal to a function of θ and φ is if both are equal to a constant λ.

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + f(r) + k^2 = -\frac{1}{\sin^2\theta}\left[\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{\Phi''}{\Phi} + g(\theta)\sin^2\theta + h(\varphi)\right] = \lambda$$

The second of these equations is

$$-\frac{1}{\sin^2\theta}\left[\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{\Phi''}{\Phi} + g(\theta)\sin^2\theta + h(\varphi)\right] = \lambda.$$

Multiply both sides by $-\sin^2\theta$.

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{\Phi''}{\Phi} + g(\theta)\sin^2\theta + h(\varphi) = -\lambda\sin^2\theta$$

Bring the first and third terms to the right side.

$$\underbrace{\frac{\Phi''}{\Phi} + h(\varphi)}_{\text{function of }\varphi} = \underbrace{-\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - g(\theta)\sin^2\theta - \lambda\sin^2\theta}_{\text{function of }\theta}$$

The only way a function of φ can be equal to a function of θ is if both are equal to another constant μ.

$$\frac{\Phi''}{\Phi} + h(\varphi) = -\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - [g(\theta) + \lambda]\sin^2\theta = \mu$$

In summary, using the method of separation of variables reduces the PDE in spherical polar coordinates to three ODEs—one in r, one in θ, and one in φ.

$$\left.\begin{aligned} \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + f(r) + k^2 &= \lambda \\ \frac{\Phi''}{\Phi} + h(\varphi) &= \mu \\ -\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - [g(\theta) + \lambda]\sin^2\theta &= \mu \end{aligned}\right\}$$

Therefore,

$$\nabla^2\psi(r,\theta,\varphi) + \left[k^2 + f(r) + \frac{1}{r^2}g(\theta) + \frac{1}{r^2\sin^2\theta}h(\varphi)\right]\psi(r,\theta,\varphi) = 0$$

is separable in spherical polar coordinates.