Exercise 9.4.7

The 1-D Schrödinger wave equation for a particle in a potential field $V = \frac{1}{2}kx^2$ is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}kx^2\psi = E\psi(x).$$

(a) Defining

$$a = \left(\frac{mk}{\hbar^2}\right)^{1/4}, \quad \lambda = \frac{2E}{\hbar}\left(\frac{m}{k}\right)^{1/2},$$

and setting $\xi = ax$, show that

$$\frac{d^2\psi(\xi)}{d\xi^2} + (\lambda - \xi^2)\psi(\xi) = 0.$$

(b) Substituting

$$\psi(\xi) = y(\xi)e^{-\xi^2/2},$$

show that $y(\xi)$ satisfies the Hermite differential equation.

Solution

Part (a)

Use the chain rule to write d/dx and d^2/dx^2 in terms of this new variable ξ.

$$\frac{d}{dx} = \frac{d\xi}{dx}\frac{d}{d\xi} = a\frac{d}{d\xi}$$

$$\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right) = a\frac{d}{d\xi}\left(a\frac{d}{d\xi}\right) = a^2\frac{d^2}{d\xi^2}$$

Consequently, the Schrödinger equation becomes

$$-\frac{\hbar^2}{2m}\left(a^2\frac{d^2}{d\xi^2}\right)\psi + \frac{1}{2}k\left(\frac{\xi}{a}\right)^2\psi = E\psi$$

$$-\frac{\hbar^2}{2m}a^2\frac{d^2\psi}{d\xi^2} + \frac{k}{2}\frac{1}{a^2}\xi^2\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{\sqrt{mk}}{\hbar}\frac{d^2\psi}{d\xi^2} + \frac{k}{2}\frac{\hbar}{\sqrt{mk}}\xi^2\psi = E\psi$$

$$-\frac{\hbar}{2}\sqrt{\frac{k}{m}}\frac{d^2\psi}{d\xi^2} + \frac{\hbar}{2}\sqrt{\frac{k}{m}}\xi^2\psi = E\psi.$$

Multiply both sides by $-\frac{2}{\hbar}\sqrt{\frac{m}{k}}$.

$$\frac{d^2\psi}{d\xi^2} - \xi^2\psi = -\frac{2E}{\hbar}\sqrt{\frac{m}{k}}\psi$$

Therefore,

$$\frac{d^2\psi(\xi)}{d\xi^2} + (\lambda - \xi^2)\psi(\xi) = 0.$$

Part (b)

Make the substitution $\psi(\xi) = y(\xi)e^{-\xi^2/2}$ in the equation.

$$\frac{d^2}{d\xi^2}[y(\xi)e^{-\xi^2/2}] + (\lambda - \xi^2)[y(\xi)e^{-\xi^2/2}] = 0$$

$$\frac{d}{d\xi}\left(e^{-\xi^2/2}y' - \xi e^{-\xi^2/2}y\right) + (\lambda - \xi^2)e^{-\xi^2/2}y = 0$$

$$e^{-\xi^2/2}y'' - \xi e^{-\xi^2/2}y' - e^{-\xi^2/2}y - \xi(-\xi)e^{-\xi^2/2}y - \xi e^{-\xi^2/2}y' + (\lambda - \xi^2)e^{-\xi^2/2}y = 0$$

$$e^{-\xi^2/2}y'' - 2\xi e^{-\xi^2/2}y' - e^{-\xi^2/2}y + \cancel{\xi^2 e^{-\xi^2/2}y} + \lambda e^{-\xi^2/2}y - \cancel{\xi^2 e^{-\xi^2/2}y} = 0$$

$$e^{-\xi^2/2}y'' - 2\xi e^{-\xi^2/2}y' + e^{-\xi^2/2}(\lambda - 1)y = 0$$

Multiply both sides by $e^{\xi^2/2}$ to obtain Hermite's differential equation.

$$y'' - 2\xi y' + (\lambda - 1)y = 0$$