Exercise 9.5.2

If Ψ is a solution of Laplace's equation, $\nabla^2\Psi = 0$, show that $\partial\Psi/\partial z$ is also a solution.

Solution

Suppose Ψ is a solution of Laplace's equation.

$$\nabla^2\Psi = 0$$

Expand the Laplacian operator in Cartesian coordinates.

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} = 0$$

Differentiate both sides with respect to z.

$$\frac{\partial}{\partial z}\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right) = 0$$

$$\frac{\partial}{\partial z}\left(\frac{\partial^2\Psi}{\partial x^2}\right) + \frac{\partial}{\partial z}\left(\frac{\partial^2\Psi}{\partial y^2}\right) + \frac{\partial}{\partial z}\left(\frac{\partial^2\Psi}{\partial z^2}\right) = 0$$

The mixed derivatives are equal and can be arranged however we like.

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial\Psi}{\partial z}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial\Psi}{\partial z}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\partial\Psi}{\partial z}\right) = 0$$

Factor $\partial\Psi/\partial z$.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial\Psi}{\partial z}\right) = 0$$

$$\nabla^2\left(\frac{\partial\Psi}{\partial z}\right) = 0$$

$\partial\Psi/\partial z$ also satisfies Laplace's equation, so it's also a solution.