

Exercise 9.6.1

Solve the wave equation, Eq. (9.89), subject to the indicated conditions.

Determine $\psi(x, t)$ given that at $t = 0$ $\psi_0(x) = \sin x$ and $\partial\psi(x)/\partial t = \cos x$.

Solution

The initial value problem to solve is as follows.

$$\begin{aligned}\psi_{tt} &= c^2\psi_{xx}, & -\infty < x < \infty, & -\infty < t < \infty \\ \psi(x, 0) &= \sin x \\ \psi_t(x, 0) &= \cos x\end{aligned}$$

Since the wave equation is over the whole line ($-\infty < x < \infty$), it can be solved by operator factorization. Bring $c^2\psi_{xx}$ to the left side.

$$\frac{\partial^2\psi}{\partial t^2} - c^2\frac{\partial^2\psi}{\partial x^2} = 0$$

Factor the operator.

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)\psi &= 0 \\ \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\psi &= 0 \\ \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial\psi}{\partial t} - c\frac{\partial\psi}{\partial x}\right) &= 0\end{aligned}$$

Let u be the quantity in the second set of parentheses.

$$\begin{aligned}\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u &= 0 \\ \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} &= 0\end{aligned}$$

As a result of factoring the operator, the wave equation has reduced to a system of first-order PDEs.

$$\left.\begin{aligned}\frac{\partial\psi}{\partial t} - c\frac{\partial\psi}{\partial x} &= u \\ \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} &= 0\end{aligned}\right\}$$

The differential of a function of two variables $h = h(x, t)$ is defined as

$$dh = \frac{\partial h}{\partial t} dt + \frac{\partial h}{\partial x} dx.$$

Divide both sides by dt to obtain the fundamental relationship between the total derivative of h and the partial derivatives of h .

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{dx}{dt} \frac{\partial h}{\partial x}$$

In light of this, the PDE for u reduces to the ODE,

$$\frac{du}{dt} = 0, \quad (1)$$

along the characteristic curves in the xt -plane that satisfy

$$\frac{dx}{dt} = c, \quad x(\xi, 0) = \xi, \quad (2)$$

where ξ is a characteristic coordinate. Integrate both sides of equation (2) with respect to t to solve for $x(\xi, t)$.

$$x = ct + \xi$$

Now integrate both sides of equation (1) with respect to t .

$$u(x, \xi) = f(\xi)$$

f is an arbitrary function of the characteristic coordinate ξ . Eliminate ξ in favor of x and t .

$$u(x, t) = f(x - ct)$$

Consequently, the PDE for ψ becomes

$$\frac{\partial \psi}{\partial t} - c \frac{\partial \psi}{\partial x} = f(x - ct).$$

It reduces to

$$\frac{d\psi}{dt} = f(x - ct) \quad (3)$$

along the characteristic curves in the xt -plane that satisfy

$$\frac{dx}{dt} = -c, \quad x(\eta, 0) = \eta, \quad (4)$$

where η is another characteristic coordinate. Integrate both sides of equation (4) with respect to t to solve for $x(\eta, t)$.

$$x = -ct + \eta$$

Now integrate both sides of equation (3) with respect to t .

$$\psi(x, \eta) = \int^t f(x - cs) ds + G(\eta)$$

G is an arbitrary function of the characteristic coordinate η . Make the substitution $r = x - cs$ in the integral.

$$\begin{aligned} \psi(x, \eta) &= \int^{x-ct} f(r) \left(-\frac{dr}{c} \right) + G(\eta) \\ &= F(x - ct) + G(\eta) \end{aligned}$$

F is the integral of $-f/c$, another arbitrary function. Therefore, since $\eta = x + ct$,

$$\psi(x, t) = F(x - ct) + G(x + ct).$$

This is the general solution of the wave equation. Now apply the initial conditions to determine F and G .

$$\begin{aligned}\psi(x, 0) &= F(x) + G(x) = \sin x \\ \psi_t(x, 0) &= -cF'(x) + cG'(x) = \cos x\end{aligned}$$

Differentiate both sides of the first equation with respect to x and multiply both sides of it by c .

$$\begin{aligned}cF'(x) + cG'(x) &= c \cos x \\ -cF'(x) + cG'(x) &= \cos x\end{aligned}$$

Add both sides of each equation to eliminate F' .

$$2cG'(x) = (c + 1) \cos x$$

Divide both sides by $2c$.

$$G'(x) = \frac{c + 1}{2c} \cos x$$

Integrate both sides with respect to x , setting the constant of integration to zero.

$$G(x) = \frac{c + 1}{2c} \sin x$$

So then

$$F(x) + G(x) = \sin x \quad \rightarrow \quad F(x) + \frac{c + 1}{2c} \sin x = \sin x \quad \rightarrow \quad F(x) = \frac{c - 1}{2c} \sin x.$$

What we have actually solved for are $F(w)$ and $G(w)$, where w is any expression we choose.

$$\begin{aligned}F(x - ct) &= \frac{c - 1}{2c} \sin(x - ct) \\ G(x + ct) &= \frac{c + 1}{2c} \sin(x + ct)\end{aligned}$$

As a result,

$$\begin{aligned}\psi(x, t) &= F(x - ct) + G(x + ct) \\ &= \frac{c - 1}{2c} \sin(x - ct) + \frac{c + 1}{2c} \sin(x + ct) \\ &= \frac{c - 1}{2c} (\sin x \cos ct - \sin ct \cos x) + \frac{c + 1}{2c} (\sin x \cos ct + \sin ct \cos x) \\ &= \left(\frac{c - 1}{2c} + \frac{c + 1}{2c} \right) \sin x \cos ct + \left(-\frac{c - 1}{2c} + \frac{c + 1}{2c} \right) \sin ct \cos x.\end{aligned}$$

Therefore,

$$\psi(x, t) = \sin x \cos ct + \frac{1}{c} \sin ct \cos x.$$