Exercise 9.2.3

Find the general solutions of the PDEs in Exercises 9.2.1 to 9.2.4.

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial z}.$$

Solution

Bring $\partial\psi/\partial z$ to the left side.

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial z} = 0 \tag{1}$$

Since ψ is a function of three variables $\psi = \psi(x, y, z)$, its differential is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy + \frac{\partial \psi}{\partial z}\,dz.$$

Dividing both sides by dx, we obtain the relationship between the total derivative of ψ and the partial derivatives of ψ.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{dy}{dx}\frac{\partial \psi}{\partial y} + \frac{dz}{dx}\frac{\partial \psi}{\partial z}$$

In light of this, equation (1) reduces to the ODE,

$$\frac{d\psi}{dx} = 0,$$

along the characteristic curves that satisfy

$$\begin{aligned} \frac{dy}{dx} &= 1, \qquad y(0,\xi) = \xi, \\ \frac{dz}{dx} &= -1, \qquad z(0,\eta) = \eta, \end{aligned}$$

where ξ and η are characteristic coordinates. Integrate both sides of each equation with respect to x.

$$\begin{aligned} \psi(x,\xi,\eta) &= f(\xi,\eta) \\ y(x,\xi) &= x + \xi \\ z(x,\eta) &= -x + \eta \end{aligned}$$

Here f is an arbitrary function of the two characteristic coordinates. Use the latter two equations to eliminate ξ and η from the first one.

$$\psi(x,y,z) = f(y - x, z + x)$$