Exercise 9.4.1

By letting the operator $\nabla^2 + k^2$ act on the general form $a_1\psi_1(x, y, z) + a_2\psi_2(x, y, z)$, show that it is linear, i.e., that $(\nabla^2 + k^2)(a_1\psi_1 + a_2\psi_2) = a_1(\nabla^2 + k^2)\psi_1 + a_2(\nabla^2 + k^2)\psi_2$.

Solution

Let the operator $\nabla^2 + k^2$ act on $a_1\psi_1(x, y, z) + a_2\psi_2(x, y, z)$.

$$\begin{aligned}
(\nabla^2 + k^2)(a_1\psi_1 + a_2\psi_2) &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)(a_1\psi_1 + a_2\psi_2) \\
&= \frac{\partial^2}{\partial x^2}(a_1\psi_1 + a_2\psi_2) + \frac{\partial^2}{\partial y^2}(a_1\psi_1 + a_2\psi_2) + \frac{\partial^2}{\partial z^2}(a_1\psi_1 + a_2\psi_2) + k^2(a_1\psi_1 + a_2\psi_2) \\
&= a_1\frac{\partial^2\psi_1}{\partial x^2} + a_2\frac{\partial^2\psi_2}{\partial x^2} + a_1\frac{\partial^2\psi_1}{\partial y^2} + a_2\frac{\partial^2\psi_2}{\partial y^2} + a_1\frac{\partial^2\psi_1}{\partial z^2} + a_2\frac{\partial^2\psi_2}{\partial z^2} + a_1k^2\psi_1 + a_2k^2\psi_2 \\
&= a_1\left(\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2} + \frac{\partial^2\psi_1}{\partial z^2} + k^2\psi_1\right) + a_2\left(\frac{\partial^2\psi_2}{\partial x^2} + \frac{\partial^2\psi_2}{\partial y^2} + \frac{\partial^2\psi_2}{\partial z^2} + k^2\psi_2\right) \\
&= a_1\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\psi_1 + a_2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\psi_2 \\
&= a_1(\nabla^2 + k^2)\psi_1 + a_2(\nabla^2 + k^2)\psi_2
\end{aligned}$$

Therefore, $\nabla^2 + k^2$ is linear.