Exercise 9.4.3

Separate variables in the Helmholtz equation in spherical polar coordinates, splitting off the radial dependence **first**. Show that your separated equations have the same form as Eqs. (9.74), (9.77), and (9.78).

Solution

The Helmholtz equation is the following PDE.

$$\nabla^2\psi + k^2\psi = 0$$

Expand the Laplacian operator in spherical polar coordinates (r, θ, φ). Here θ is the angle from the polar axis.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} + k^2\psi = 0$$

Separate variables by assuming a product solution of the form $\psi(r, \theta, \varphi) = R(r)F(\theta, \varphi)$ and substituting it into the equation.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}[R(r)F(\theta,\varphi)]\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}[R(r)F(\theta,\varphi)]\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}[R(r)F(\theta,\varphi)] + k^2[R(r)F(\theta,\varphi)] = 0$$

$$\frac{F}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{R}{r^2\sin^2\theta}\frac{\partial^2 F}{\partial\varphi^2} + k^2RF = 0$$

$$\frac{F}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2\sin^2\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{\partial^2 F}{\partial\varphi^2}\right] + k^2RF = 0$$

Divide both sides by RF.

$$\frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{r^2F\sin^2\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{\partial^2 F}{\partial\varphi^2}\right] + k^2 = 0$$

Multiply both sides by r^2 and bring the second term to the right side.

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2r^2}_{\text{function of } r} = \underbrace{-\frac{1}{F\sin^2\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{\partial^2 F}{\partial\varphi^2}\right]}_{\text{function of } \theta \text{ and } \varphi}$$

The only way a function of r can be equal to a function of θ and φ is if both are equal to a constant λ.

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2r^2 = -\frac{1}{F\sin^2\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{\partial^2 F}{\partial\varphi^2}\right] = \lambda$$

The equation that F satisfies is

$$-\frac{1}{F\sin^2\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right)+\frac{\partial^2 F}{\partial\varphi^2}\right]=\lambda.$$

Assume a product solution of the form $F(\theta,\varphi)=\Theta(\theta)\Phi(\varphi)$ and substitute it into the equation.

$$-\frac{1}{[\Theta(\theta)\Phi(\varphi)]\sin^2\theta}\left\{\sin\theta\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}[\Theta(\theta)\Phi(\varphi)]\right]+\frac{\partial^2}{\partial\varphi^2}[\Theta(\theta)\Phi(\varphi)]\right\}=\lambda$$

$$-\frac{1}{\Theta\Phi\sin^2\theta}\left[\Phi\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\Theta\Phi''(\varphi)\right]=\lambda$$

$$-\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{\Phi''}{\Phi\sin^2\theta}=\lambda$$

Multiply both sides by $\sin^2\theta$ and bring the first term over to the right side.

$$\underbrace{-\frac{\Phi''}{\Phi}}_{\text{function of }\varphi}=\underbrace{\lambda\sin^2\theta+\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{\text{function of }\theta}$$

The only way a function of φ can be equal to a function of θ is if both are equal to another constant μ.

$$-\frac{\Phi''}{\Phi}=\lambda\sin^2\theta+\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=\mu$$

In summary, using the method of separation of variables reduces the Helmholtz equation in spherical polar coordinates to three ODEs—one in r, one in θ, and one in φ.

$$\left.\begin{aligned}\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+k^2r^2=\lambda\\ -\frac{\Phi''}{\Phi}=\mu\\ \lambda\sin^2\theta+\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=\mu\end{aligned}\right\}$$

The first ODE leads to Eq. (9.78) by bringing λ to the left side and multiplying both sides by R/r^2. The second ODE leads to Eq. (9.74) by setting $\mu=m^2$ and multiplying both sides by -1. The third ODE leads to Eq. (9.77) by setting $\mu=m^2$, bringing μ to the left side, and multiplying both sides by $\Theta/\sin^2\theta$.