Exercise 9.6.3

Solve the wave equation, Eq. (9.89), subject to the indicated conditions.

Determine $\psi(x,t)$ given that at $t=0$ $\psi_0(x)$ is a single square-wave pulse as defined below, and the initial time derivative of ψ is zero.

$$\psi_0(x) = 0,\ |x| > a/2, \quad \psi_0(x) = 1/a,\ |x| < a/2.$$

Solution

The initial value problem to solve is as follows.

$$\psi_{tt} = c^2\psi_{xx}, \quad -\infty < x < \infty,\ -\infty < t < \infty$$

$$\psi(x,0) = \psi_0(x) = \begin{cases} \dfrac{1}{a} & |x| < \dfrac{a}{2} \\ 0 & |x| > \dfrac{a}{2} \end{cases}$$

$$\psi_t(x,0) = 0$$

Since the wave equation is over the whole line ($-\infty < x < \infty$), it can be solved by operator factorization. Bring $c^2\psi_{xx}$ to the left side.

$$\frac{\partial^2\psi}{\partial t^2} - c^2\frac{\partial^2\psi}{\partial x^2} = 0$$

Factor the operator.

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)\psi = 0$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\psi = 0$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial\psi}{\partial t} - c\frac{\partial\psi}{\partial x}\right) = 0$$

Let u be the quantity in the second set of parentheses.

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0$$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

As a result of factoring the operator, the wave equation has reduced to a system of first-order PDEs.

$$\left.\begin{aligned} \frac{\partial\psi}{\partial t} - c\frac{\partial\psi}{\partial x} &= u \\ \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} &= 0 \end{aligned}\right\}$$

The differential of a function of two variables $h = h(x,t)$ is defined as

$$dh = \frac{\partial h}{\partial t}\,dt + \frac{\partial h}{\partial x}\,dx.$$

Divide both sides by dt to obtain the fundamental relationship between the total derivative of h and the partial derivatives of h.

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{dx}{dt}\frac{\partial h}{\partial x}$$

In light of this, the PDE for u reduces to the ODE,

$$\frac{du}{dt} = 0, \tag{1}$$

along the characteristic curves in the xt-plane that satisfy

$$\frac{dx}{dt} = c, \quad x(\xi, 0) = \xi, \tag{2}$$

where ξ is a characteristic coordinate. Integrate both sides of equation (2) with respect to t to solve for $x(\xi, t)$.

$$x = ct + \xi$$

Now integrate both sides of equation (1) with respect to t.

$$u(x, \xi) = f(\xi)$$

f is an arbitrary function of the characteristic coordinate ξ. Eliminate ξ in favor of x and t.

$$u(x, t) = f(x - ct)$$

Consequently, the PDE for ψ becomes

$$\frac{\partial \psi}{\partial t} - c\frac{\partial \psi}{\partial x} = f(x - ct).$$

It reduces to

$$\frac{d\psi}{dt} = f(x - ct) \tag{3}$$

along the characteristic curves in the xt-plane that satisfy

$$\frac{dx}{dt} = -c, \quad x(\eta, 0) = \eta, \tag{4}$$

where η is another characteristic coordinate. Integrate both sides of equation (4) with respect to t to solve for $x(\eta, t)$.

$$x = -ct + \eta$$

Now integrate both sides of equation (3) with respect to t.

$$\psi(x, \eta) = \int^t f(x - cs)\, ds + G(\eta)$$

G is an arbitrary function of the characteristic coordinate η. Make the substitution $r = x - cs$ in the integral.

$$\begin{aligned}\psi(x, \eta) &= \int^{x-ct} f(r)\left(-\frac{dr}{c}\right) + G(\eta) \\ &= F(x - ct) + G(\eta)\end{aligned}$$

F is the integral of $-f/c$, another arbitrary function. Therefore, since $\eta = x + ct$,

$$\psi(x,t) = F(x - ct) + G(x + ct).$$

This is the general solution of the wave equation. Now apply the initial conditions to determine F and G.

$$\begin{aligned} \psi(x,0) &= F(x) + G(x) = \psi_0(x) \\ \psi_t(x,0) &= -cF'(x) + cG'(x) = 0 \end{aligned}$$

Differentiate both sides of the first equation with respect to x and multiply both sides of it by c.

$$\begin{aligned} cF'(x) + cG'(x) &= c\psi_0'(x) \\ -cF'(x) + cG'(x) &= 0 \end{aligned}$$

Add both sides of each equation to eliminate F'.

$$2cG'(x) = c\psi_0'(x)$$

Divide both sides by $2c$.

$$G'(x) = \frac{1}{2}\psi_0'(x)$$

Integrate both sides with respect to x, setting the constant of integration to zero.

$$G(x) = \frac{1}{2}\psi_0(x)$$

So then

$$F(x) + G(x) = \psi_0(x) \quad \rightarrow \quad F(x) + \frac{1}{2}\psi_0(x) = \psi_0(x) \quad \rightarrow \quad F(x) = \frac{1}{2}\psi_0(x).$$

What we have actually solved for are $F(w)$ and $G(w)$, where w is any expression we choose.

$$\begin{aligned} F(x - ct) &= \frac{1}{2}\psi_0(x - ct) \\ G(x + ct) &= \frac{1}{2}\psi_0(x + ct) \end{aligned}$$

As a result,

$$\begin{aligned} \psi(x,t) &= F(x - ct) + G(x + ct) \\ &= \frac{1}{2}\psi_0(x - ct) + \frac{1}{2}\psi_0(x + ct) \\ &= \frac{1}{2}[\psi_0(x - ct) + \psi_0(x + ct)]. \end{aligned}$$

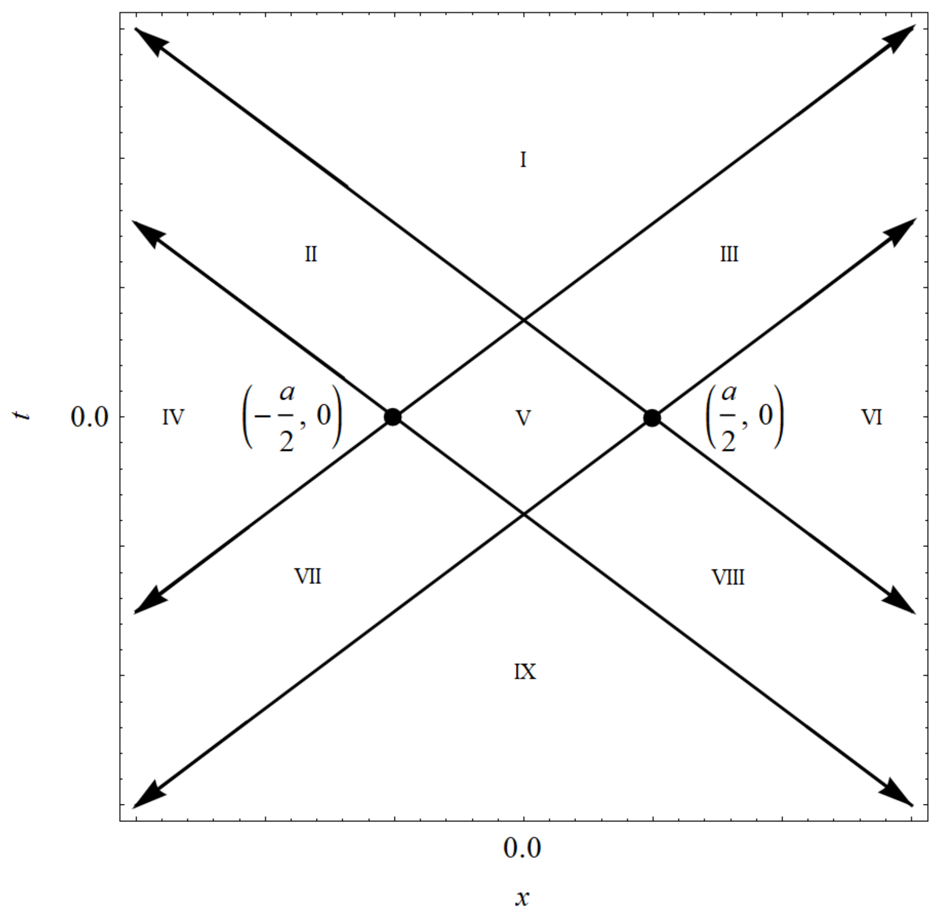
Note that

$$\psi_0(x-ct) = \begin{cases} \dfrac{1}{a} & |x-ct| < \dfrac{a}{2} \\ 0 & |x-ct| > \dfrac{a}{2} \end{cases} = \begin{cases} \dfrac{1}{a} & -\dfrac{a}{2} < x-ct < \dfrac{a}{2} \\ 0 & x-ct < -\dfrac{a}{2} \\ 0 & x-ct > \dfrac{a}{2} \end{cases}$$

and

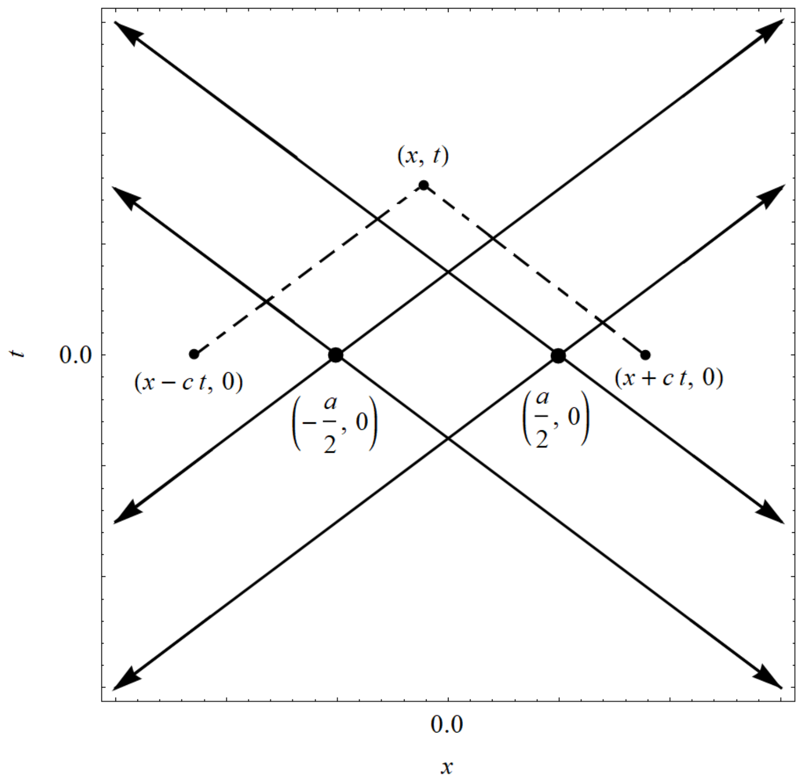
$$\psi_0(x+ct) = \begin{cases} \dfrac{1}{a} & |x+ct| < \dfrac{a}{2} \\ 0 & |x+ct| > \dfrac{a}{2} \end{cases} = \begin{cases} \dfrac{1}{a} & -\dfrac{a}{2} < x+ct < \dfrac{a}{2} \\ 0 & x+ct < -\dfrac{a}{2} \\ 0 & x+ct > \dfrac{a}{2} \end{cases}.$$

Depending what region in the xt-plane the point (x,t) is chosen, $\psi(x,t)$ will be different. These regions are obtained by drawing characteristic lines with slopes $\pm c$ through $\left(-\frac{a}{2},0\right)$ and $\left(\frac{a}{2},0\right)$, the boundaries of where the initial condition is nonzero.



Region I

Suppose the point (x, t) is chosen in region I.



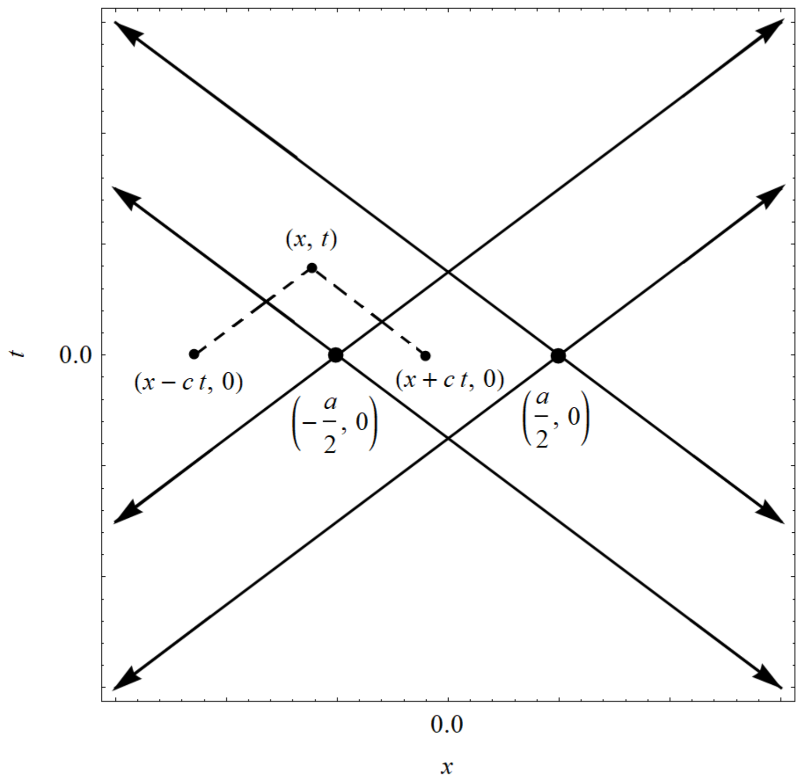
In this case $x - ct < -\frac{a}{2}$ and $x + ct > \frac{a}{2}$, so

$$\begin{aligned}\psi(x,t) &= \frac{1}{2}[\psi_0(x-ct) + \psi_0(x+ct)] \\ &= \frac{1}{2}(0+0) \\ &= 0.\end{aligned}$$

This formula is valid for $|x| < ct - \frac{a}{2}$.

Region II

Suppose the point (x, t) is chosen in region II.



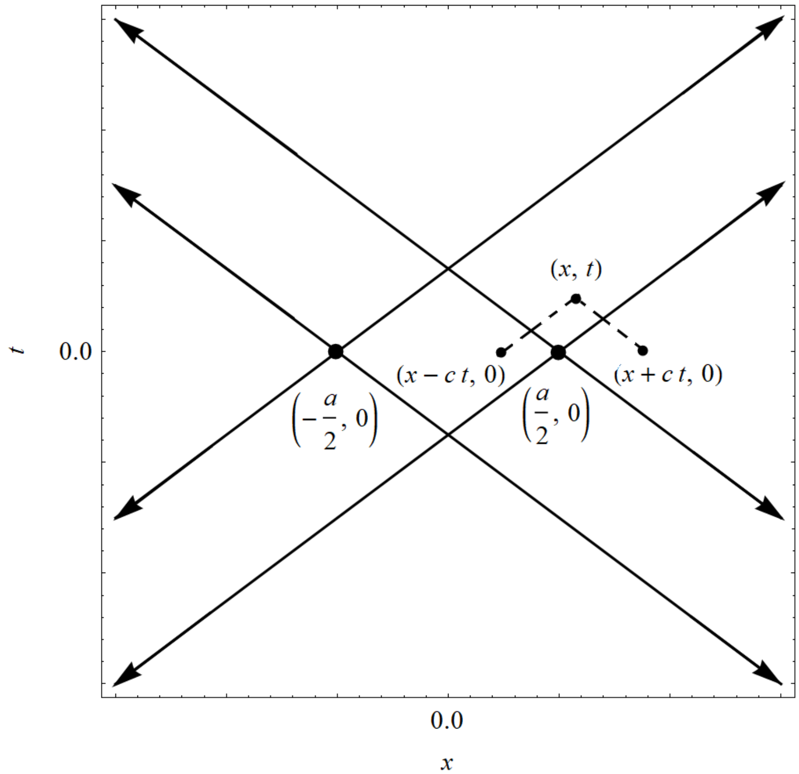
In this case $x - ct < -\frac{a}{2}$ and $-\frac{a}{2} < x + ct < \frac{a}{2}$, so

$$\begin{aligned}\psi(x,t) &= \frac{1}{2}[\psi_0(x-ct)+\psi_0(x+ct)] \\ &= \frac{1}{2}\left(0+\frac{1}{a}\right) \\ &= \frac{1}{2a}.\end{aligned}$$

This formula is valid for $|\frac{a}{2} - ct| < -x < \frac{a}{2} + ct$.

Region III

Suppose the point (x, t) is chosen in region III.



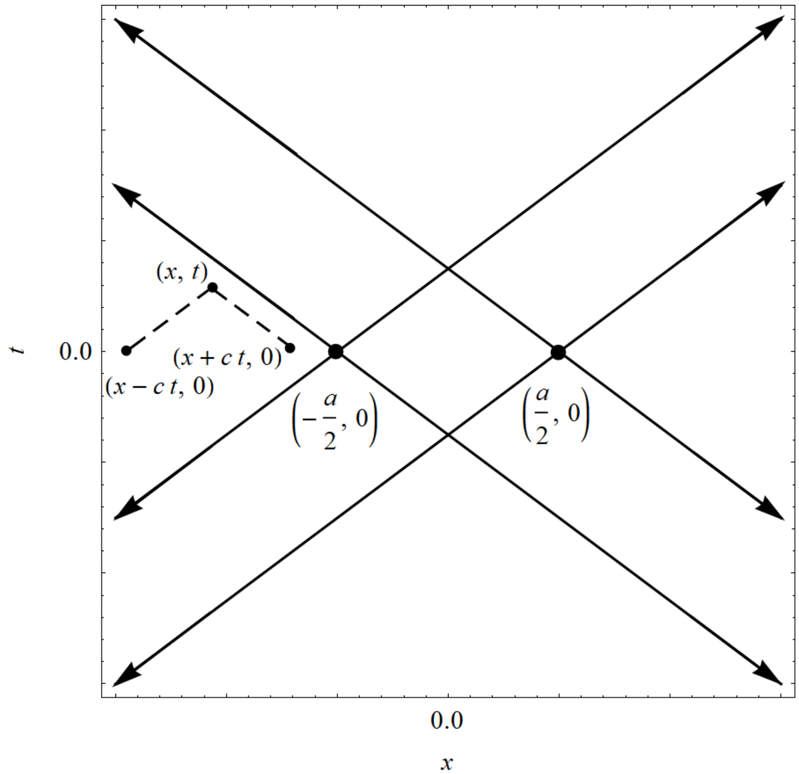
In this case $-\frac{a}{2} < x - ct < \frac{a}{2}$ and $x + ct > \frac{a}{2}$, so

$$\begin{aligned}\psi(x,t) &= \frac{1}{2}[\psi_0(x-ct) + \psi_0(x+ct)] \\ &= \frac{1}{2}\left(\frac{1}{a} + 0\right) \\ &= \frac{1}{2a}.\end{aligned}$$

This formula is valid for $|\frac{a}{2} - ct| < x < \frac{a}{2} + ct$.

Region IV

Suppose the point (x, t) is chosen in region IV.



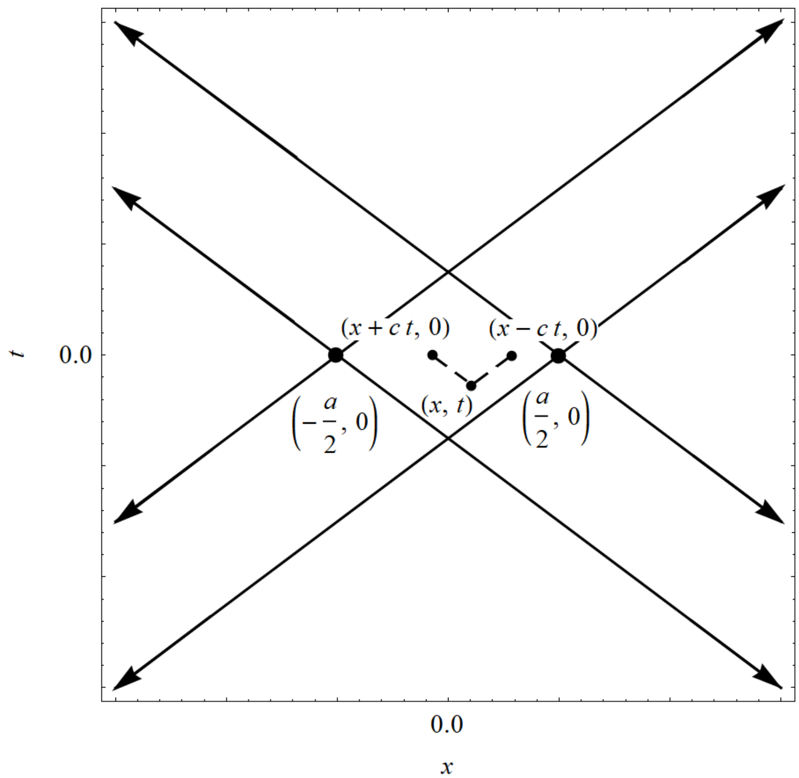
In this case $x - ct < -\frac{a}{2}$ and $x + ct < -\frac{a}{2}$, so

$$\begin{aligned}\psi(x,t) &= \frac{1}{2}[\psi_0(x-ct) + \psi_0(x+ct)] \\ &= \frac{1}{2}(0+0) \\ &= 0.\end{aligned}$$

This formula is valid for $-x > \frac{a}{2} + c|t|$.

Region V

Suppose the point (x, t) is chosen in region V.



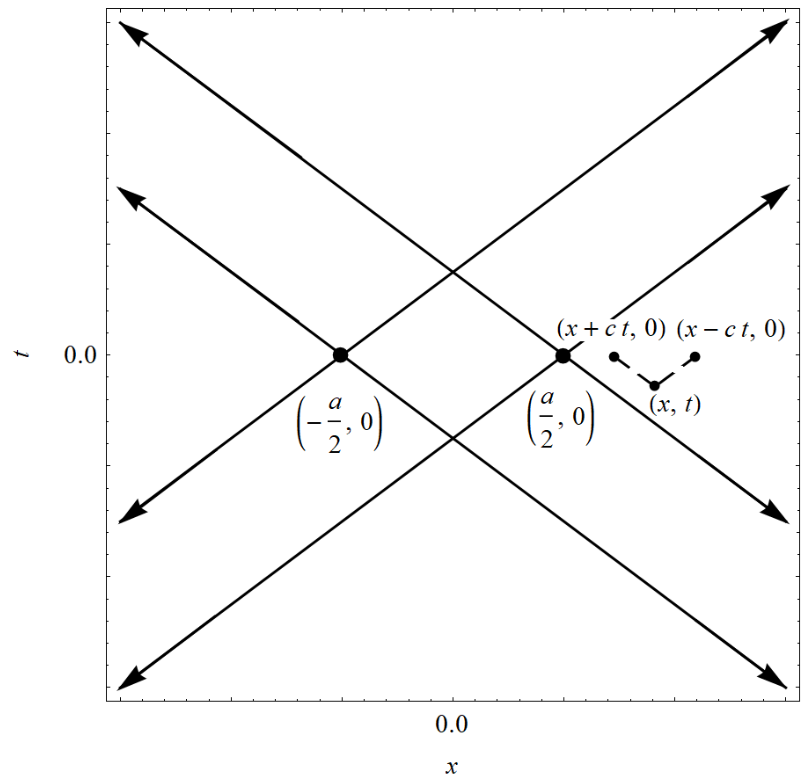
In this case $-\frac{a}{2} < x - ct < \frac{a}{2}$ and $-\frac{a}{2} < x + ct < \frac{a}{2}$, so

$$\begin{aligned}\psi(x,t) &= \frac{1}{2}[\psi_0(x-ct)+\psi_0(x+ct)] \\ &= \frac{1}{2}\left(\frac{1}{a}+\frac{1}{a}\right) \\ &= \frac{1}{a}.\end{aligned}$$

This formula is valid for $|x| < \frac{a}{2} - c|t|$.

Region VI

Suppose the point (x, t) is chosen in region VI.



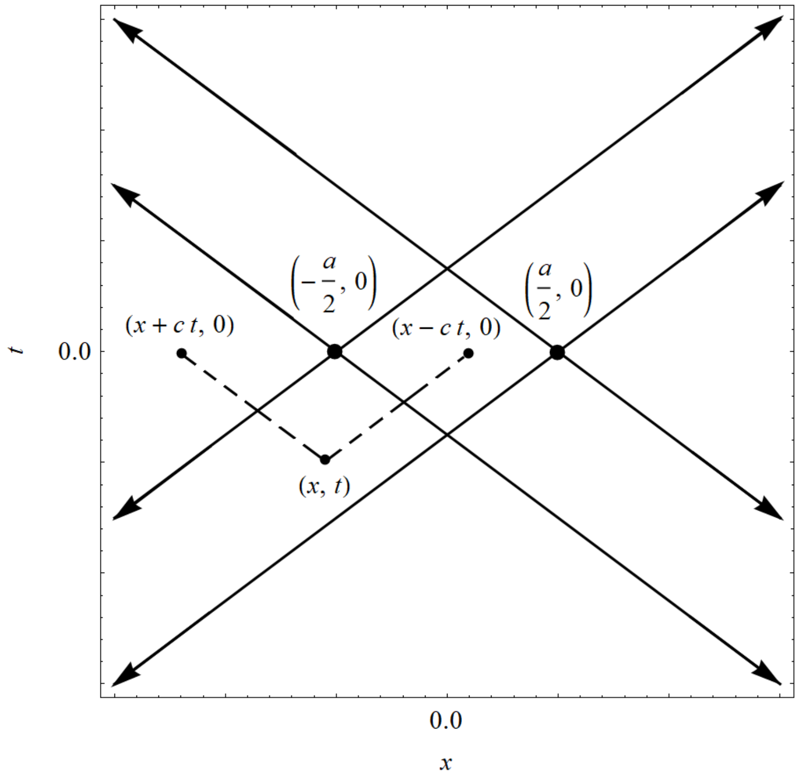
In this case $x - ct > \frac{a}{2}$ and $x + ct > \frac{a}{2}$, so

$$\begin{aligned}\psi(x,t) &= \frac{1}{2}[\psi_0(x-ct) + \psi_0(x+ct)] \\ &= \frac{1}{2}(0+0) \\ &= 0.\end{aligned}$$

This formula is valid for $x > \frac{a}{2} + c|t|$.

Region VII

Suppose the point (x, t) is chosen in region VII.



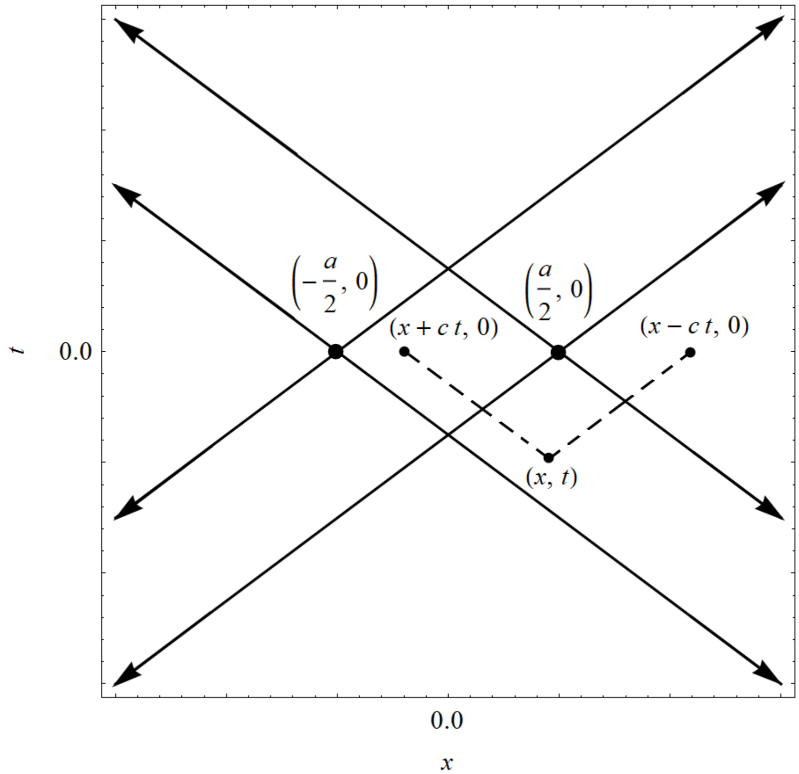
In this case $-\frac{a}{2} < x - ct < \frac{a}{2}$ and $x + ct < -\frac{a}{2}$, so

$$\begin{aligned}\psi(x,t) &= \frac{1}{2}[\psi_0(x-ct)+\psi_0(x+ct)] \\ &= \frac{1}{2}\left(\frac{1}{a}+0\right) \\ &= \frac{1}{2a}.\end{aligned}$$

This formula is valid for $|\frac{a}{2} + ct| < -x < \frac{a}{2} - ct$.

Region VIII

Suppose the point (x, t) is chosen in region VIII.



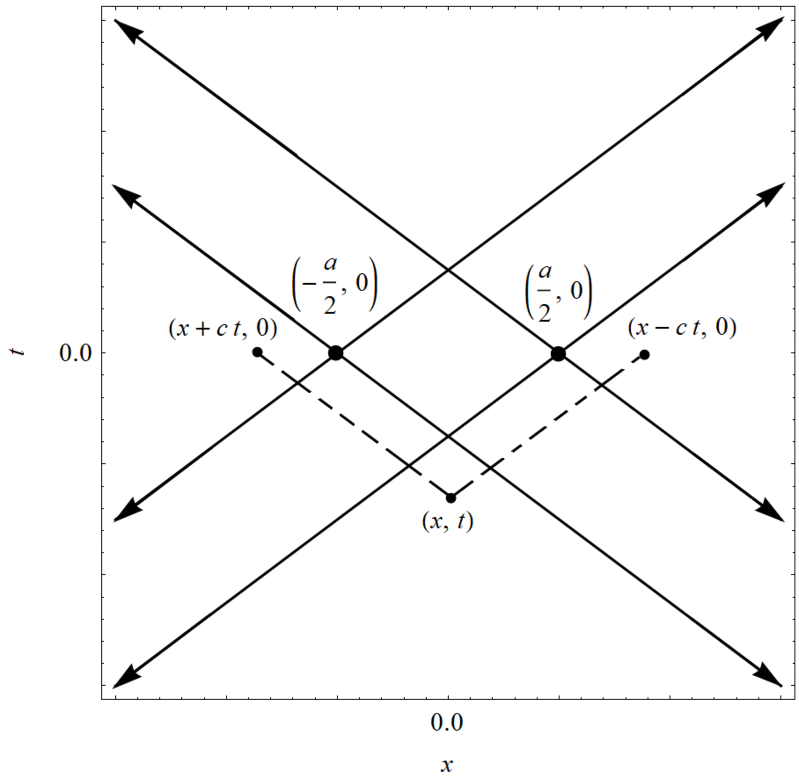
In this case $x - ct > \frac{a}{2}$ and $-\frac{a}{2} < x + ct < \frac{a}{2}$, so

$$\begin{aligned}\psi(x,t) &= \frac{1}{2}[\psi_0(x-ct) + \psi_0(x+ct)] \\ &= \frac{1}{2}\left(0 + \frac{1}{a}\right) \\ &= \frac{1}{2a}.\end{aligned}$$

This formula is valid for $|\frac{a}{2} + ct| < x < \frac{a}{2} - ct$.

Region IX

Suppose the point (x, t) is chosen in region IX.



In this case $x - ct > \frac{a}{2}$ and $x + ct < -\frac{a}{2}$, so

$$\begin{aligned} \psi(x,t) &= \frac{1}{2}[\psi_0(x-ct) + \psi_0(x+ct)] \\ &= \frac{1}{2}(0+0) \\ &= 0. \end{aligned}$$

This formula is valid for $|x| < -ct - \frac{a}{2}$.

Some of the formulas we found can be combined by forming unions of the regions and using absolute value signs. The union of regions I and IX, for example, results in the following formula for $\psi(x, t)$.

$$\psi(x,t) = 0 \quad \text{if } |x| < c|t| - \frac{a}{2}$$

The union of regions IV and VI results in the following formula for $\psi(x, t)$.

$$\psi(x,t) = 0 \quad \text{if } |x| > \frac{a}{2} + c|t|$$

The union of regions II, III, VII, and VIII results in the following formula for $\psi(x, t)$.

$$\psi(x,t) = \frac{1}{2a} \quad \text{if } \left|\frac{a}{2} - c|t|\right| < |x| < \frac{a}{2} + c|t|$$

Therefore,

$$\psi(x,t) = \begin{cases} \frac{1}{a} & \text{if } |x| < \frac{a}{2} - c|t| \\ 0 & \text{if } |x| < c|t| - \frac{a}{2} \\ 0 & \text{if } |x| > \frac{a}{2} + c|t| \\ \frac{1}{2a} & \text{if } \left|\frac{a}{2} - c|t|\right| < |x| < \frac{a}{2} + c|t| \end{cases}.$$

The solution in each part of the xt-plane is labelled by color.