Exercise 9.7.2

Separate variables in the thermal diffusion equation of Exercise 9.7.1 in circular cylindrical coordinates. Assume that you can neglect end effects and take $T = T(\rho, t)$.

Solution

The thermal diffusion equation of Exercise 9.7.1 is

$$\frac{\partial T}{\partial t} = K\nabla^2 T.$$

Expand the Laplacian operator in circular cylindrical coordinates (ρ, φ, z).

$$\frac{\partial T(\rho,t)}{\partial t} = K\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial T(\rho,t)}{\partial \rho}\right) + \underbrace{\frac{1}{\rho^2}\frac{\partial^2 T(\rho,t)}{\partial \varphi^2}}_{=0} + \underbrace{\frac{\partial^2 T(\rho,t)}{\partial z^2}}_{=0}\right]$$

T is only a function of ρ and t, so the angular derivatives vanish.

$$\frac{\partial T}{\partial t} = \frac{K}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial T}{\partial \rho}\right)$$

The equation of heat conduction is linear and homogeneous, so the method of separation of variables can be applied to solve it. Assume a product solution of the form $T(\rho, t) = P(\rho)\Theta(t)$ and substitute it into the PDE.

$$\frac{\partial}{\partial t}[P(\rho)\Theta(t)] = \frac{K}{\rho}\frac{\partial}{\partial \rho}\left[\rho\frac{\partial}{\partial \rho}[P(\rho)\Theta(t)]\right]$$

$$P\frac{d\Theta}{dt} = \Theta\frac{K}{\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)$$

Divide both sides by $KP(\rho)\Theta(t)$. (The final answer for $T(\rho, t)$ will be the same regardless which side K is on.)

$$\underbrace{\frac{1}{K\Theta}\frac{d\Theta}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)}_{\text{function of } \rho}$$

The only way a function of t can be equal to a function of ρ is if both are equal to a constant λ.

$$\frac{1}{K\Theta}\frac{d\Theta}{dt} = \frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) = \lambda$$

As a result of applying the method of separation of variables, the equation of conduction has reduced to two ODEs—one in ρ and one in t.

$$\left.\begin{aligned} \frac{1}{K\Theta}\frac{d\Theta}{dt} &= \lambda \\ \frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) &= \lambda \end{aligned}\right\}$$

Solve the first ODE for Θ.

$$\frac{d\Theta}{dt} = K\lambda\Theta$$

The general solution is written in terms of the exponential function.

$$\Theta(t) = C_1 e^{K\lambda t}$$

In order for $T(\rho, t)$ to remain bounded as $t \to \infty$, we require that λ be either zero or negative. Suppose first that λ is zero: $\lambda = 0$. The ODE for P becomes

$$\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) = 0.$$

Multiply both sides by ρP.

$$\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) = 0.$$

Integrate both sides with respect to ρ.

$$\rho\frac{dP}{d\rho} = C_2$$

Divide both sides by ρ.

$$\frac{dP}{d\rho} = \frac{C_2}{\rho}$$

Integrate both sides with respect to ρ once more.

$$P(\rho) = C_2 \ln\rho + C_3$$

Note that this is the steady-state temperature profile in a cylindrical geometry. With two boundary conditions, one could determine the constants, C_2 and C_3. Suppose secondly that λ is negative: $\lambda = -\alpha^2$. The ODE for P becomes

$$\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) = -\alpha^2.$$

Multiply both sides by $\rho^2 P$.

$$\rho\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) = -\alpha^2\rho^2 P$$

Use the product rule to expand the left side.

$$\rho\left(\rho\frac{d^2P}{d\rho^2} + \frac{dP}{d\rho}\right) = -\alpha^2\rho^2 P$$

The radial equation is thus

$$\rho^2\frac{d^2P}{d\rho^2} + \rho\frac{dP}{d\rho} + \alpha^2\rho^2 P = 0,$$

which is known as the Bessel equation of order zero. Its general solution is written in terms of J_0 and Y_0, the Bessel functions of the first and second kind, respectively.

$$P(\rho) = C_4 J_0(\alpha\rho) + C_5 Y_0(\alpha\rho)$$