Exercise 1

Suppose that u_1 and u_2 are solutions of (1). Show that $c_1u_1 + c_2u_2$ is also a solution, where c_1 and c_2 are constants. (This shows that (1) is a linear equation.)

Solution

Equation (1) is

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0. \tag{1}$$

Suppose that u_1 and u_2 are solutions of this PDE. Then they satisfy

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u_2}{\partial t} + \frac{\partial u_2}{\partial x} = 0.$$

Check to see if $c_1u_1 + c_2u_2$ is also a solution by plugging it into equation (1) and using the fact that derivatives are linear operators.

$$\begin{aligned}
0 &\stackrel{?}{=} \frac{\partial}{\partial t}(c_1u_1 + c_2u_2) + \frac{\partial}{\partial x}(c_1u_1 + c_2u_2) \\
&\stackrel{?}{=} c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} + c_1\frac{\partial u_1}{\partial x} + c_2\frac{\partial u_2}{\partial x} \\
&\stackrel{?}{=} c_1\left(\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x}\right) + c_2\left(\frac{\partial u_2}{\partial t} + \frac{\partial u_2}{\partial x}\right) \\
&\stackrel{?}{=} c_1(0) + c_2(0) \\
&= 0
\end{aligned}$$

Therefore, $c_1u_1 + c_2u_2$ is a solution of equation (1) as well.