Exercise 10

Consider the equation $a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0$, where a, b are nonzero constants.

(a) What is the equation saying about the directional derivative of u?

(b) Determine the characteristic curves.

(c) Solve the equation using the method of characteristic curves.

Solution

Notice that the left side can be written as the dot product of two vectors, $\langle a, b\rangle$ and $\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right\rangle$.

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0$$

$$\langle a, b\rangle \cdot \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right\rangle = 0$$

The PDE says that the directional derivative of u in the direction of $\langle a, b\rangle$ is zero at any point. This means u varies in the direction perpendicular to $\langle a, b\rangle$, that is, in the direction of $\langle b, -a\rangle$.

$$u(x, y) = f(bx - ay)$$

The differential of a two-dimensional function $g = g(x, y)$ is given by

$$dg = \frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy.$$

Dividing both sides by dx yields the fundamental relationship between the total derivative of g and its partial derivatives.

$$\frac{dg}{dx} = \frac{\partial g}{\partial x} + \frac{dy}{dx}\frac{\partial g}{\partial y}$$

Comparing this to the PDE,

$$\frac{\partial u}{\partial x} + \frac{b}{a}\frac{\partial u}{\partial y} = 0,$$

we see that along the (characteristic) curves in the xy-plane defined by

$$\frac{dy}{dx} = \frac{b}{a} \tag{1}$$

the PDE reduces to the ODE,

$$\frac{du}{dx} = 0. \tag{2}$$

Solve equation (1), using ξ for the characteristic coordinate.

$$y = \frac{b}{a}x + \xi \quad \rightarrow \quad \xi = y - \frac{b}{a}x$$

Then solve equation (2) by integrating both sides with respect to x.

$$u(x, \xi) = F(\xi)$$

Now that u is known, change back to the original variables.

$$u(x, y) = F\left(y - \frac{b}{a}x\right)$$