Exercise 11

In Exercises 11-14, (a) solve the given equation by the method of characteristic curves, and (b) check your answer by plugging it back into the equation.

$$\frac{\partial u}{\partial x} + x^2 \frac{\partial u}{\partial y} = 0.$$

Solution

The differential of a two-dimensional function $g = g(x, y)$ is given by

$$dg = \frac{\partial g}{\partial x}\, dx + \frac{\partial g}{\partial y}\, dy.$$

Dividing both sides by dx yields the fundamental relationship between the total derivative of g and its partial derivatives.

$$\frac{dg}{dx} = \frac{\partial g}{\partial x} + \frac{dy}{dx}\frac{\partial g}{\partial y}$$

Comparing this to the PDE, we see that along the (characteristic) curves in the xy-plane defined by

$$\frac{dy}{dx} = x^2 \tag{1}$$

the PDE reduces to the ODE,

$$\frac{du}{dx} = 0. \tag{2}$$

Solve equation (1), using ξ for the characteristic coordinate.

$$y = \frac{1}{3}x^3 + \xi \quad \rightarrow \quad \xi = y - \frac{1}{3}x^3$$

Then solve equation (2) by integrating both sides with respect to x.

$$u(x, \xi) = f(\xi)$$

Here f is an arbitrary function. Now that u is known, change back to the original variables.

$$u(x, y) = f\left(y - \frac{1}{3}x^3\right)$$

Compute the first derivatives to check the solution.

$$\frac{\partial u}{\partial x} = f'\left(y - \frac{1}{3}x^3\right) \cdot \frac{\partial}{\partial x}\left(y - \frac{1}{3}x^3\right) = f'\left(y - \frac{1}{3}x^3\right) \cdot (-x^2) = -x^2 f'$$

$$\frac{\partial u}{\partial y} = f'\left(y - \frac{1}{3}x^3\right) \cdot \frac{\partial}{\partial y}\left(y - \frac{1}{3}x^3\right) = f'\left(y - \frac{1}{3}x^3\right) \cdot (1) = f'$$

As a result,

$$\frac{\partial u}{\partial x} + x^2 \frac{\partial u}{\partial y} = -x^2 f' + x^2 f' = 0.$$