Exercise 14

In Exercises 11-14, (a) solve the given equation by the method of characteristic curves, and (b) check your answer by plugging it back into the equation.

$$e^{x^2}\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0.$$

Solution

Divide both sides by e^{x^2}.

$$\frac{\partial u}{\partial x} + xe^{-x^2}\frac{\partial u}{\partial y} = 0$$

The differential of a two-dimensional function $g = g(x, y)$ is given by

$$dg = \frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy.$$

Dividing both sides by dx yields the fundamental relationship between the total derivative of g and its partial derivatives.

$$\frac{dg}{dx} = \frac{\partial g}{\partial x} + \frac{dy}{dx}\frac{\partial g}{\partial y}$$

Comparing this to the PDE, we see that along the (characteristic) curves in the xy-plane defined by

$$\frac{dy}{dx} = xe^{-x^2} \tag{1}$$

the PDE reduces to the ODE,

$$\frac{du}{dx} = 0. \tag{2}$$

Solve equation (1), making the substitution $v = -x^2$ $(dv = -2x\,dx)$ and using ξ for the characteristic coordinate.

$$y = \int xe^{-x^2}\,dx = \int e^v\left(-\frac{dv}{2}\right) = -\frac{1}{2}\int e^v\,dv = -\frac{1}{2}e^v + \xi = -\frac{1}{2}e^{-x^2} + \xi \quad \rightarrow \quad \xi = y + \frac{1}{2}e^{-x^2}$$

Then solve equation (2) by integrating both sides with respect to x.

$$u(x, \xi) = f(\xi)$$

Here f is an arbitrary function. Now that u is known, change back to the original variables.

$$u(x, y) = f\left(y + \frac{1}{2}e^{-x^2}\right)$$

Compute the first derivatives to check the solution.

$$\frac{\partial u}{\partial x} = f'\left(y + \frac{1}{2}e^{-x^2}\right) \cdot \frac{\partial}{\partial x}\left(y + \frac{1}{2}e^{-x^2}\right) = f'\left(y + \frac{1}{2}e^{-x^2}\right) \cdot (-xe^{-x^2}) = -xe^{-x^2}f'$$

$$\frac{\partial u}{\partial y} = f'\left(y + \frac{1}{2}e^{-x^2}\right) \cdot \frac{\partial}{\partial y}\left(y + \frac{1}{2}e^{-x^2}\right) = f'\left(y + \frac{1}{2}e^{-x^2}\right) \cdot (1) = f'$$

As a result,

$$e^{x^2}\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = -xf' + xf' = 0.$$