# Exercise 2

(a) Which solution of (1) is equal to $xe^{-x^2}$ on the $x$-axis?

(b) Plot the solution as a function of $x$ and $t$, and describe the image of the $x$-axis.

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### Solution

Equation (1) is

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0. \tag{1}$$

The solution that's equal to $xe^{-x^2}$ on the $x$-axis satisfies the following initial value problem.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \ -\infty < t < \infty$$

$$u(x, 0) = xe^{-x^2}$$

Make the change of variables, $\alpha = x + t$ and $\beta = x - t$, and use the chain rule to write the derivatives in terms of these new variables.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(1) = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(-1) = \frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}$$

The PDE then becomes

$$\begin{aligned} 0 &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \\ &= \left(\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}\right) + \left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) \\ &= 2\frac{\partial u}{\partial \alpha}. \end{aligned}$$

Divide both sides by 2.

$$\frac{\partial u}{\partial \alpha} = 0$$

Integrate both sides partially with respect to $\alpha$ to get $u$.

$$u(\alpha, \beta) = f(\beta)$$

Here $f$ is an arbitrary function. Now that the general solution to the PDE is known, change back to the original variables.

$$u(x, t) = f(x - t)$$

To determine $f$, use the initial condition.

$$u(x, 0) = f(x) = xe^{-x^2}$$

What this actually means is that $f(w) = we^{-w^2}$, where $w$ is any expression, so

$$f(x - t) = (x - t)e^{-(x-t)^2}.$$

Therefore,

$$u(x, t) = (x - t)e^{-(x-t)^2}.$$

Below is a plot of $u(x,t)$ versus $x$ at several moments in time.