Exercise 8

In Exercises 5-8, derive the general solution of the given equation by using an appropriate change of variables, as we did in Example 3.

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = u, \quad a, b \neq 0.$$

Solution

Make the change of variables, $\alpha = ax + bt$ and $\beta = ax - bt$, and use the chain rule to write the derivatives in terms of these new variables.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha}(a) + \frac{\partial u}{\partial \beta}(a) = a\frac{\partial u}{\partial \alpha} + a\frac{\partial u}{\partial \beta}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = \frac{\partial u}{\partial \alpha}(b) + \frac{\partial u}{\partial \beta}(-b) = b\frac{\partial u}{\partial \alpha} - b\frac{\partial u}{\partial \beta}$$

The PDE then becomes

$$\begin{aligned} u &= a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} \\ &= a\left(b\frac{\partial u}{\partial \alpha} - b\frac{\partial u}{\partial \beta}\right) + b\left(a\frac{\partial u}{\partial \alpha} + a\frac{\partial u}{\partial \beta}\right) \\ &= 2ab\frac{\partial u}{\partial \alpha}. \end{aligned}$$

Subtract both sides by u.

$$2ab\frac{\partial u}{\partial \alpha} - u = 0$$

Divide both sides by $2ab$.

$$\frac{\partial u}{\partial \alpha} - \frac{1}{2ab}u = 0 \tag{1}$$

This is a first-order linear differential equation, so it can be solved by using an integrating factor.

$$I = \exp\left(\int^{\alpha} \frac{-1}{2ab}\, ds\right) = e^{-\alpha/(2ab)}$$

Multiply both sides of equation (1) by I.

$$e^{-\alpha/(2ab)}\frac{\partial u}{\partial \alpha} - \frac{1}{2ab}e^{-\alpha/(2ab)}u = 0$$

Use the product rule.

$$\frac{\partial}{\partial \alpha}\left[e^{-\alpha/(2ab)}u\right] = 0$$

Integrate both sides partially with respect to α.

$$e^{-\alpha/(2ab)}u = f(\beta)$$

Here f is an arbitrary function. Multiply both sides by $e^{\alpha/(2ab)}$.

$$u(\alpha, \beta) = e^{\alpha/(2ab)}f(\beta)$$

Now that the general solution to the PDE is known, change back to the original variables.

$$u(x, t) = e^{(ax+bt)/(2ab)}f(ax - bt)$$