Exercise 9

(a) Find the solution in Exercise 5 that is equal to $\frac{1}{1+x^2}$ along the x-axis.

(b) Plot the graph of the solution as a function of x and t.

(c) In what direction is the waveform moving as t increases?

Solution

The solution in Exercise 5 that is equal to $\frac{1}{1+x^2}$ along the x-axis satisfies the following initial value problem.

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \ -\infty < t < \infty$$

$$u(x,0) = \frac{1}{1+x^2}$$

Make the change of variables, $\alpha = x + t$ and $\beta = x - t$, and use the chain rule to write the derivatives in terms of these new variables.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(1) = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(-1) = \frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}$$

The PDE then becomes

$$\begin{aligned} 0 &= \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} \\ &= \left(\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}\right) - \left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) \\ &= -2\frac{\partial u}{\partial \beta}. \end{aligned}$$

Divide both sides by -2.

$$\frac{\partial u}{\partial \beta} = 0$$

Integrate both sides partially with respect to β to get u.

$$u(\alpha, \beta) = f(\alpha)$$

Here f is an arbitrary function. Now that the general solution to the PDE is known, change back to the original variables.

$$u(x,t) = f(x+t)$$

To determine f, apply the initial condition.

$$u(x,0) = f(x) = \frac{1}{1+x^2}$$

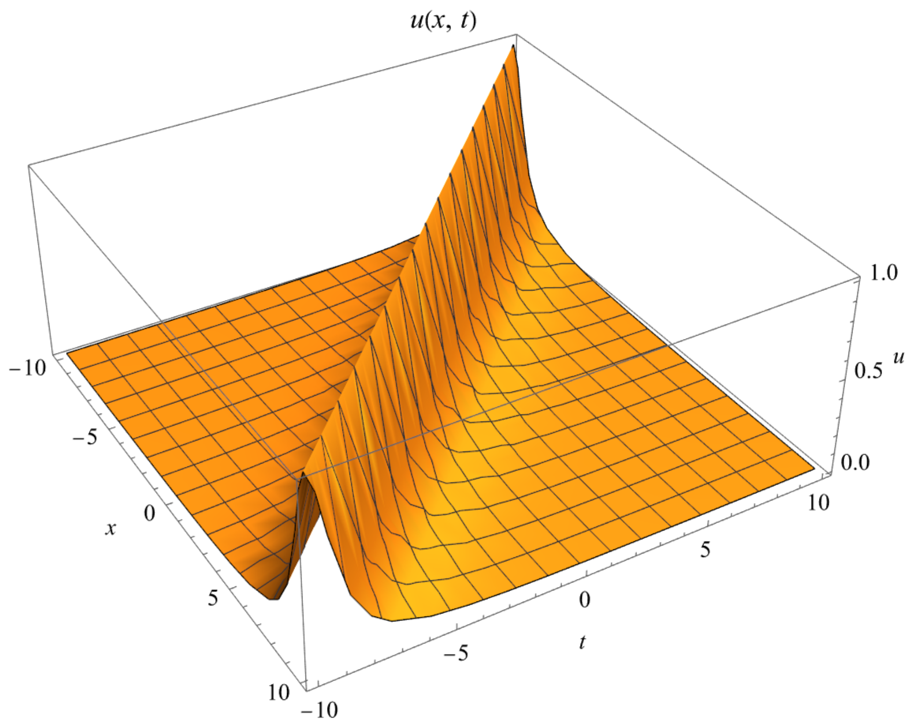
What this actually means is that $f(w) = \frac{1}{1+w^2}$, where w is any expression, so

$$f(x+t) = \frac{1}{1+(x+t)^2}.$$

Therefore,

$$u(x,t) = \frac{1}{1+(x+t)^2}.$$

Below is a plot of this solution versus x and t.



Notice that as t increases, the x-coordinate of the waveform's peak decreases. In other words, the waveform is moving in the negative x-direction.