Exercise 14

Referring to Figure 8, explain why the displacements in frames 2, 4, 6, and 8 appear to obey the end conditions

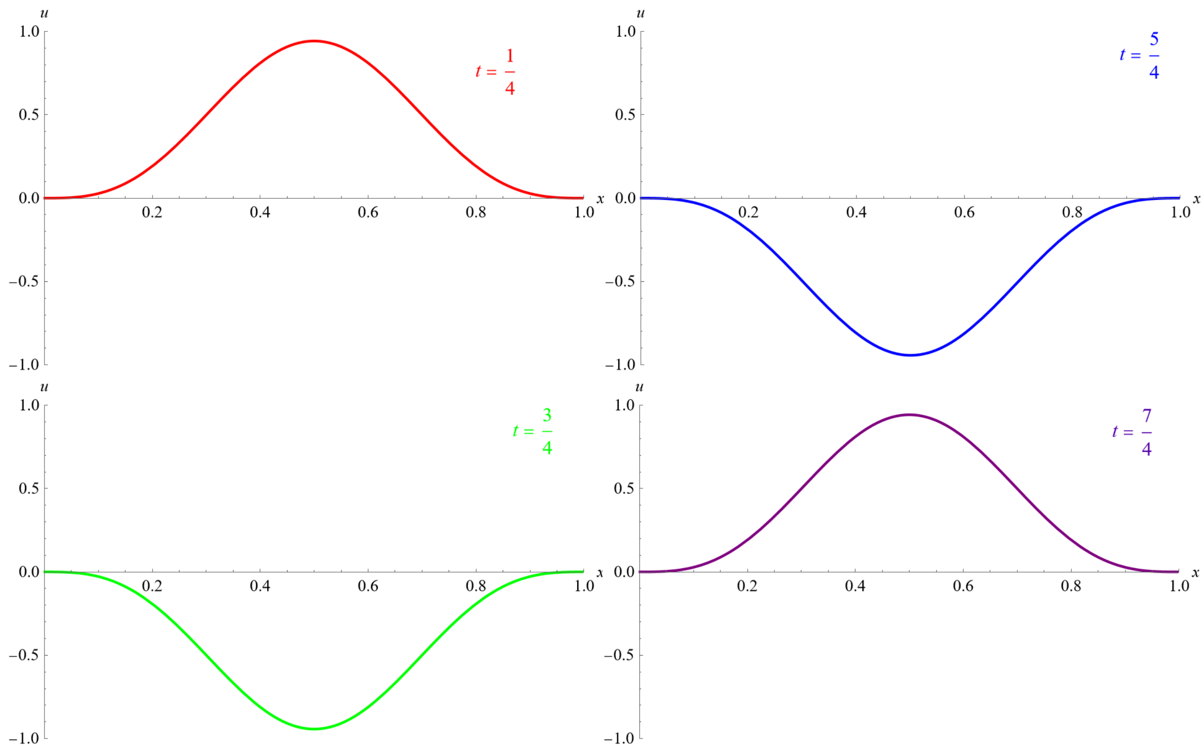
$$u(0,t) = 0 = u(1,t) \quad \text{and} \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0.$$

Solution

Figure 8 shows the graph of

$$u(x,t) = \sin \pi x \cos \pi t - \frac{1}{2}\sin 2\pi x \cos 2\pi t + \frac{1}{3}\sin 3\pi x \cos 3\pi t, \quad 0 < x < 1,$$

a solution to the wave equation on the interval $0 < x < L$ with fixed ends, versus x at several times. The graphs in frames 2, 4, 6, and 8 are shown below.



Plugging in $x = 0$ and $x = 1$, we see that the amplitude is zero at these points at all times.

$$u(0,t) = \sin 0 \cos \pi t - \frac{1}{2}\sin 0 \cos 2\pi t + \frac{1}{3}\sin 0 \cos 3\pi t = 0$$

$$u(1,t) = \sin \pi \cos \pi t - \frac{1}{2}\sin 2\pi \cos 2\pi t + \frac{1}{3}\sin 3\pi \cos 3\pi t = 0$$

The graphs do seem to be zero and have zero slope at $x = 0$ and $x = 1$, and they occur at $t = 1/4$, $t = 3/4$, $t = 5/4$, and $t = 7/4$.

To examine the slope of the graph, take the derivative of u with respect to x.

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}(\sin \pi x \cos \pi t - \frac{1}{2}\sin 2\pi x \cos 2\pi t + \frac{1}{3}\sin 3\pi x \cos 3\pi t) \\ &= \pi \cos \pi x \cos \pi t - \pi \cos 2\pi x \cos 2\pi t + \pi \cos 3\pi x \cos 3\pi t\end{aligned}$$

Then plug in $t = (2n+1)/4$, where $n = 0, 1, 2, \ldots$.

$$\begin{aligned}\frac{\partial u}{\partial x}\left(x, \frac{2n+1}{4}\right) &= \pi \cos \pi x \cos\left[\pi\left(\frac{2n+1}{4}\right)\right] - \pi \cos 2\pi x \cos\left[2\pi\left(\frac{2n+1}{4}\right)\right] + \pi \cos 3\pi x \cos\left[3\pi\left(\frac{2n+1}{4}\right)\right] \\ &= \pi \cos \pi x \cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right) - \pi \cos 2\pi x \cos\left(n\pi + \frac{\pi}{2}\right) + \pi \cos 3\pi x \cos\left(\frac{3n\pi}{2} + \frac{3\pi}{4}\right) \\ &= \pi \cos \pi x\left(\cos\frac{n\pi}{2}\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\sin\frac{\pi}{4}\right) - \pi \cos 2\pi x\left(\cos n\pi \underbrace{\cos\frac{\pi}{2}}_{=0} - \underbrace{\sin n\pi}_{=0}\sin\frac{\pi}{2}\right) \\ &\qquad + \pi \cos 3\pi x\left(\cos\frac{3n\pi}{2}\cos\frac{3\pi}{4} - \sin\frac{3n\pi}{2}\sin\frac{3\pi}{4}\right) \\ &= \pi \cos \pi x\left(\frac{1}{\sqrt{2}}\cos\frac{n\pi}{2} - \frac{1}{\sqrt{2}}\sin\frac{n\pi}{2}\right) + \pi \cos 3\pi x\left(-\frac{1}{\sqrt{2}}\cos\frac{3n\pi}{2} - \frac{1}{\sqrt{2}}\sin\frac{3n\pi}{2}\right) \\ &= \frac{\pi}{\sqrt{2}}\cos \pi x\left(\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right) - \frac{\pi}{\sqrt{2}}\cos 3\pi x\left(\cos\frac{3n\pi}{2} + \sin\frac{3n\pi}{2}\right)\end{aligned}$$

Set $x = 0$ and use the sum-to-product formulas.

$$\begin{aligned}\frac{\partial u}{\partial x}\left(0, \frac{2n+1}{4}\right) &= \frac{\pi}{\sqrt{2}}\cos 0\left(\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right) - \frac{\pi}{\sqrt{2}}\cos 0\left(\cos\frac{3n\pi}{2} + \sin\frac{3n\pi}{2}\right) \\ &= \frac{\pi}{\sqrt{2}}\left(\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right) - \frac{\pi}{\sqrt{2}}\left(\cos\frac{3n\pi}{2} + \sin\frac{3n\pi}{2}\right) \\ &= \frac{\pi}{\sqrt{2}}\left(\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2} - \cos\frac{3n\pi}{2} - \sin\frac{3n\pi}{2}\right) \\ &= -\frac{\pi}{\sqrt{2}}\left[\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right) + \left(\sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2}\right)\right] \\ &= -\frac{\pi}{\sqrt{2}}\left[-2\sin\left(\frac{\frac{3n\pi}{2} + \frac{n\pi}{2}}{2}\right)\sin\left(\frac{\frac{3n\pi}{2} - \frac{n\pi}{2}}{2}\right) + 2\sin\left(\frac{\frac{3n\pi}{2} + \frac{n\pi}{2}}{2}\right)\cos\left(\frac{\frac{3n\pi}{2} - \frac{n\pi}{2}}{2}\right)\right] \\ &= -\frac{\pi}{\sqrt{2}}\left(-2\underbrace{\sin n\pi}_{=0}\sin\frac{n\pi}{2} + 2\underbrace{\sin n\pi}_{=0}\cos\frac{n\pi}{2}\right) \\ &= 0\end{aligned}$$

Set $x = 1$ and use the sum-to-product formulas.

$$\begin{aligned}
\frac{\partial u}{\partial x}\left(1, \frac{2n+1}{4}\right) &= \frac{\pi}{\sqrt{2}} \cos \pi \left(\cos \frac{n\pi}{2} - \sin \frac{n\pi}{2}\right) - \frac{\pi}{\sqrt{2}} \cos 3\pi \left(\cos \frac{3n\pi}{2} + \sin \frac{3n\pi}{2}\right) \\
&= -\frac{\pi}{\sqrt{2}} \left(\cos \frac{n\pi}{2} - \sin \frac{n\pi}{2}\right) + \frac{\pi}{\sqrt{2}} \left(\cos \frac{3n\pi}{2} + \sin \frac{3n\pi}{2}\right) \\
&= \frac{\pi}{\sqrt{2}} \left(-\cos \frac{n\pi}{2} + \sin \frac{n\pi}{2} + \cos \frac{3n\pi}{2} + \sin \frac{3n\pi}{2}\right) \\
&= \frac{\pi}{\sqrt{2}} \left[\left(\cos \frac{3n\pi}{2} - \cos \frac{n\pi}{2}\right) + \left(\sin \frac{3n\pi}{2} + \sin \frac{n\pi}{2}\right)\right] \\
&= \frac{\pi}{\sqrt{2}} \left[-2 \sin \left(\frac{\frac{3n\pi}{2} + \frac{n\pi}{2}}{2}\right) \sin \left(\frac{\frac{3n\pi}{2} - \frac{n\pi}{2}}{2}\right) + 2 \sin \left(\frac{\frac{3n\pi}{2} + \frac{n\pi}{2}}{2}\right) \cos \left(\frac{\frac{3n\pi}{2} - \frac{n\pi}{2}}{2}\right)\right] \\
&= \frac{\pi}{\sqrt{2}} \left(-2 \underbrace{\sin n\pi}_{=0} \sin \frac{n\pi}{2} + 2 \underbrace{\sin n\pi}_{=0} \cos \frac{n\pi}{2}\right) \\
&= 0
\end{aligned}$$

Therefore, the slope of the graphs at $x = 0$ and $x = 1$ when $t = 1/4$, $t = 3/4$, $t = 5/4$, and $t = 7/4$.