Exercise 22

(a) Solve the wave equation with $c = 1$, boundary conditions (3), (4) with $L = 1$, and initial data $f(x) = \frac{1}{2}\sin 2\pi x + \frac{1}{4}\sin 4\pi x$, $g(x) = 0$.

(b) Plot several snapshots of the string. How many fixed points do you see in the interval $0 < x < 1$? Justify your answer.

Solution

The general solution to the wave equation on a finite interval with fixed ends and arbitrary initial shape and zero velocity,

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \ -\infty < t < \infty \\
&u(x,0) = \frac{1}{2}\sin 2\pi x + \frac{1}{4}\sin 4\pi x \\
&\frac{\partial u}{\partial t}(x,0) = 0 \\
&u(0,t) = 0 \\
&u(L,t) = 0,
\end{aligned}$$

is (to be derived in later chapters)

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}.$$

To determine the constants A_n, set $t = 0$ and substitute the given function for $u(x,0)$.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

$$\frac{1}{2}\sin 2\pi x + \frac{1}{4}\sin 4\pi x = A_1\sin\frac{\pi x}{L} + A_2\sin\frac{2\pi x}{L} + A_3\sin\frac{3\pi x}{L} + \cdots$$

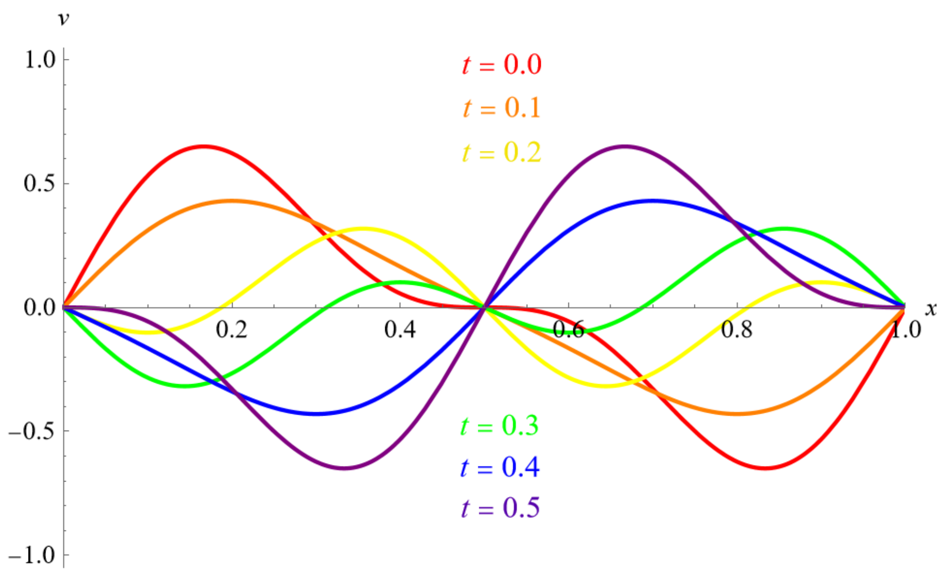
Then match the coefficients on both sides.

$$\begin{aligned}
A_1 &= 0 \\
A_2 &= \frac{1}{2} \\
A_4 &= \frac{1}{4} \\
&\vdots \\
A_n &= 0, \quad n \neq 2, 4
\end{aligned}$$

Therefore, the general solution that satisfies the initial conditions is

$$\begin{aligned} u(x,t) &= A_2 \sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L} + A_4 \sin\frac{4\pi x}{L}\cos\frac{4\pi ct}{L} \\ &= \frac{1}{2}\sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L} + \frac{1}{4}\sin\frac{4\pi x}{L}\cos\frac{4\pi ct}{L}. \end{aligned}$$

Below is a plot of u versus x over $0 < x < 1$ at several times with $c = 1$ and $L = 1$.



Notice that u is zero at $x = \frac{1}{2}$ at all times. Plug in $c = 1$ and $L = 1$ and $x = \frac{1}{2}$ in the solution.

$$u\left(\frac{1}{2}, t\right) = \frac{1}{2}\sin\pi\cos 2\pi t + \frac{1}{4}\sin 2\pi\cos 4\pi t = 0$$

This is zero regardless of what t is.