

Exercise 23

In this exercise, we consider how a given point on a vibrating string moves with time. Consider the solution of Exercise 15 with $c = L = 1$. Fix $x = x_0$ and plot $u(x_0, t)$ as a function of t , for $0 < t < 20$, for $x_0 = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}$. Observe that in each case we get a cosine wave and that all the curves are identical except for a scaling factor.

Solution

The initial boundary value problem from Exercise 15 is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad -\infty < t < \infty$$

$$u(x, 0) = \sin \frac{2\pi x}{L}$$

$$\frac{\partial u}{\partial t}(x, 0) = 0$$

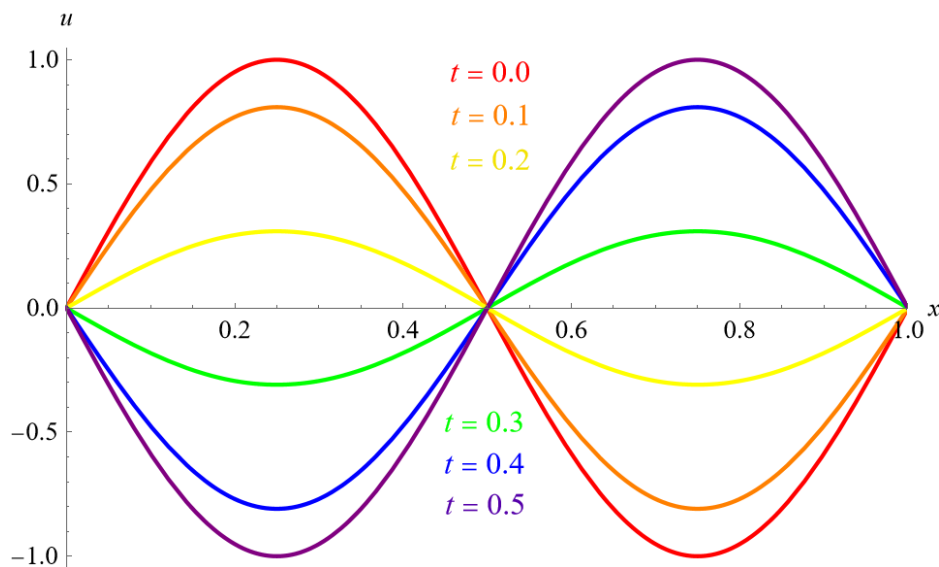
$$u(0, t) = 0$$

$$u(L, t) = 0,$$

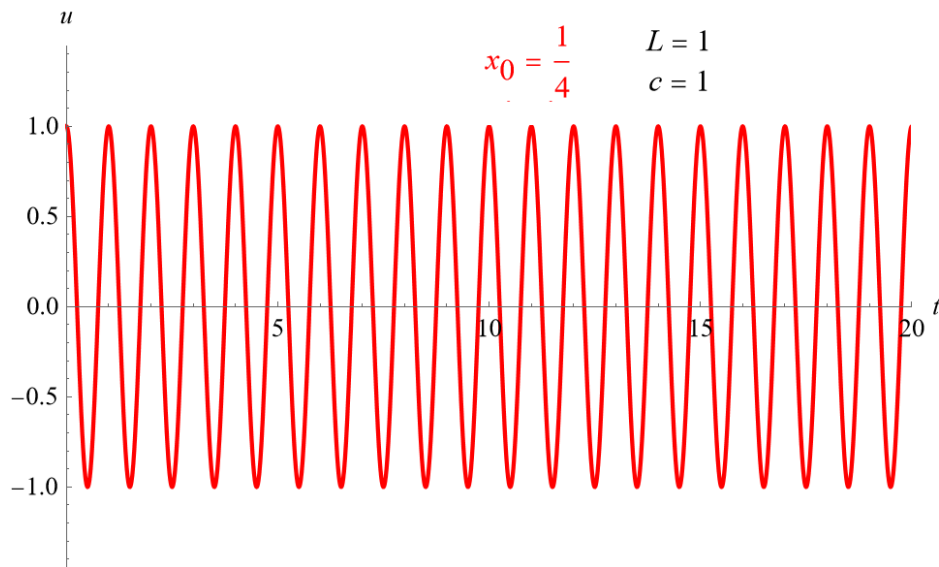
and its solution is

$$u(x, t) = \sin \frac{2\pi x}{L} \cos \frac{2\pi ct}{L}.$$

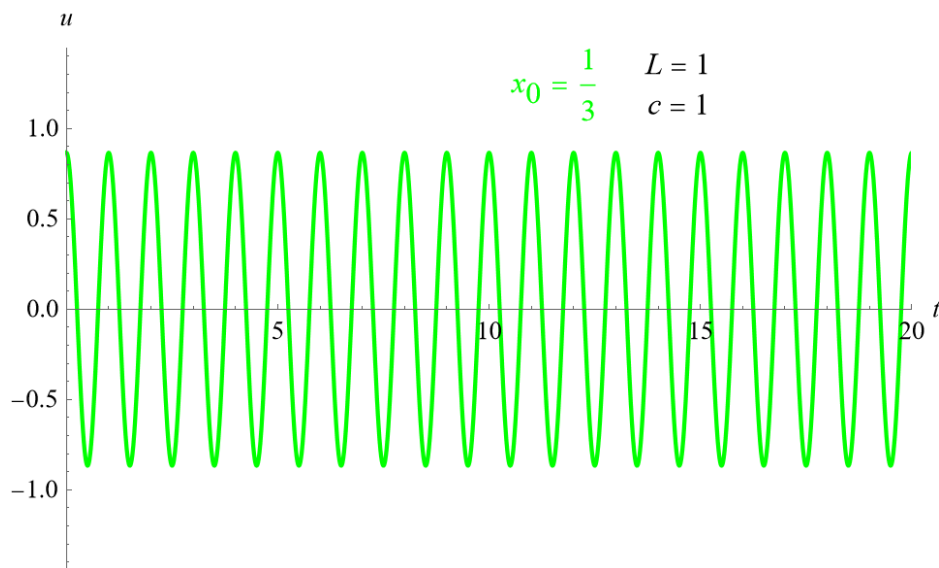
Below is a plot of u versus x over $0 < x < 1$ at several times with $c = 1$ and $L = 1$.



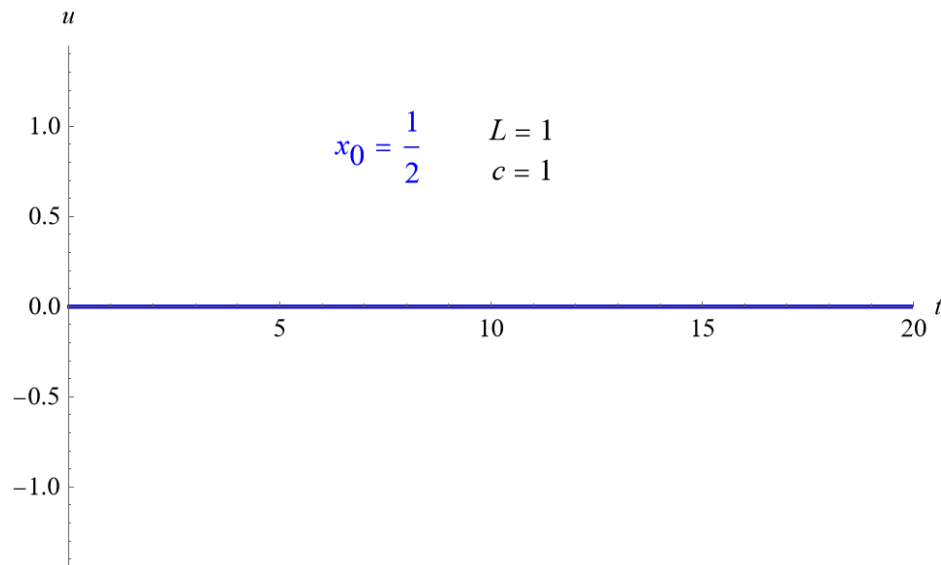
Below are plots of $u(x_0, t)$ versus t with $c = 1$ and $L = 1$ for $x_0 = \frac{1}{4}$,



for $x_0 = \frac{1}{3}$,



for $x_0 = \frac{1}{2}$,



and for $x_0 = \frac{3}{4}$.

