Exercise 3

Let F and G be arbitrary differentiable functions of one variable. Show that $u(x,t) = F(x+ct) + G(x-ct)$ is a solution to the wave equation (1), provided that F and G are sufficiently smooth. (This solution will be derived in the next exercise.)

Solution

The aim is to solve the wave equation on the whole line for all time.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \ -\infty < t < \infty$$

Since $-\infty < x < \infty$, the method of operator factorization can be applied to solve it.

$$\begin{aligned} 0 &= \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} \\ &= \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u \\ &= \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u \end{aligned}$$

If we set

$$v = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x},$$

then the previous equation becomes

$$\begin{aligned} 0 &= \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) v \\ &= \frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x}. \end{aligned}$$

In other words, the wave equation is equivalent to the following system of first-order equations.

$$\begin{cases} \dfrac{\partial u}{\partial t} - c\dfrac{\partial u}{\partial x} = v(x,t) \\ \dfrac{\partial v}{\partial t} + c\dfrac{\partial v}{\partial x} = 0 \end{cases}$$

Solve the second equation for v first by using the method of characteristics. Along the curves in the xt-plane defined by

$$\frac{dx}{dt} = c, \quad x(\xi, 0) = 0, \tag{1}$$

where ξ is a characteristic coordinate, the PDE reduces to an ODE.

$$\frac{dv}{dt} = 0 \tag{2}$$

Solve equation (1) by integrating both sides with respect to t.

$$x = ct + \xi \quad \to \quad \xi = x - ct$$

Then solve equation (2) for v.

$$v(\xi, t) = g(\xi)$$

Here g represents an arbitrary function. Now that v is known, change back to the original variables.

$$v(x, t) = g(x - ct)$$

As a result, the PDE for u becomes

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = g(x - ct).$$

Use the method of characteristics again to solve it. Along the curves in the xt-plane defined by

$$\frac{dx}{dt} = -c, \quad x(\eta, 0) = \eta, \tag{3}$$

where η is another characteristic coordinate, the PDE reduces to an ODE.

$$\frac{du}{dt} = g(x - ct) \tag{4}$$

Solve equation (3) by integrating both sides with respect to t.

$$x = -ct + \eta \quad \to \quad \eta = x + ct$$

Then solve equation (4), using this formula to eliminate x.

$$\frac{du}{dt} = g(-2ct + \eta)$$

Integrate both sides with respect to t.

$$u(\eta, t) = \int^t g(-2cs + \eta)\, ds + F(\eta)$$

The integral of an arbitrary function is another arbitrary function.

$$u(\eta, t) = G(-2ct + \eta) + F(\eta)$$

Now that u is known, change back to the original variables.

$$u(x, t) = G(x - ct) + F(x + ct)$$

This is the general solution to the wave equation. Two arbitrary functions are present, and two initial conditions are necessary to determine them. Check to see that the wave equation is satisfied by finding the second derivatives of $u(x, t)$.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left[G(x - ct) + F(x + ct)\right] = G'(x - ct) \cdot (-c) + F'(x + ct) \cdot (c) = -cG'(x - ct) + cF'(x + ct)$$

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}[-cG'(x - ct) + cF'(x + ct)] = -cG''(x - ct) \cdot (-c) + cF''(x + ct) \cdot (c) \\ &= c^2 G''(x - ct) + c^2 F''(x + ct)\end{aligned}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[G(x - ct) + F(x + ct)\right] = G'(x - ct) \cdot (1) + F'(x + ct) \cdot (1) = G'(x - ct) + F'(x + ct)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}[G'(x - ct) + F'(x + ct)] = G''(x - ct) \cdot (1) + F''(x + ct) \cdot (1) = G''(x - ct) + F''(x + ct)$$

Indeed,

$$\frac{\partial^2 u}{\partial t^2} = c^2 G''(x - ct) + c^2 F''(x + ct) = c^2 \frac{\partial^2 u}{\partial x^2}$$

for any two twice-differentiable functions, F and G.