Exercise 4

(a) Use the change of variables $\alpha = x + ct$, $\beta = x - ct$ to transform the wave equation (1) into $\frac{\partial^2 u}{\partial\alpha\partial\beta} = 0$. (You should assume that $\frac{\partial^2 u}{\partial\alpha\partial\beta} = \frac{\partial^2 u}{\partial\beta\partial\alpha}$.)

(b) Integrate the equation with respect to α to obtain $\frac{\partial u}{\partial\beta} = g(\beta)$, where g is an arbitrary function.

(c) Integrate with respect to β to arrive at $u = F(\alpha) + G(\beta)$, where F is an arbitrary function and G is an antiderivative of g.

(d) Derive the solution given in Exercise 3.

Solution

The aim is to solve the wave equation on the whole line for all time.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty,\ -\infty < t < \infty$$

Bring both terms to the left side.

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Comparing this to the general form of a second-order PDE,
$Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + fu = g$, we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $f = 0$, and $g = 0$. The characteristic curves for this second-order PDE satisfy

$$\begin{aligned} \frac{dx}{dt} &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \\ &= \frac{\pm\sqrt{-4(1)(-c^2)}}{2(1)} \\ &= \pm c. \end{aligned}$$

The two families of real characteristic curves are

$$x = ct + C_1 \quad \text{and} \quad x = -ct + C_2.$$

The constants of integration are essentially the names of the family members.

$$C_1 = x - ct$$

$$C_2 = x + ct$$

The wave equation can be simplified tremendously by making the change of variables, $\alpha = x + ct$ and $\beta = x - ct$. Use the chain rule to write the derivatives in terms of these new variables.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = \frac{\partial u}{\partial \alpha}(c) + \frac{\partial u}{\partial \beta}(-c) = c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \left(\frac{\partial \alpha}{\partial t}\frac{\partial}{\partial \alpha} + \frac{\partial \beta}{\partial t}\frac{\partial}{\partial \beta}\right)\left(\frac{\partial u}{\partial t}\right) = \left(c\frac{\partial}{\partial \alpha} - c\frac{\partial}{\partial \beta}\right)\left(c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta}\right) = c^2\frac{\partial^2 u}{\partial \alpha^2} - 2c^2\frac{\partial^2 u}{\partial \alpha \partial \beta} + c^2\frac{\partial^2 u}{\partial \beta^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(1) = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial \alpha}{\partial x}\frac{\partial}{\partial \alpha} + \frac{\partial \beta}{\partial x}\frac{\partial}{\partial \beta}\right)\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right)\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) = \frac{\partial^2 u}{\partial \alpha^2} + 2\frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta^2}$$

Substitute these formula into the wave equation.

$$\begin{aligned} 0 &= \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} \\ &= \left(c^2\frac{\partial^2 u}{\partial \alpha^2} - 2c^2\frac{\partial^2 u}{\partial \alpha \partial \beta} + c^2\frac{\partial^2 u}{\partial \beta^2}\right) - c^2\left(\frac{\partial^2 u}{\partial \alpha^2} + 2\frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta^2}\right) \\ &= -4c^2\frac{\partial^2 u}{\partial \alpha \partial \beta} \end{aligned}$$

Divide both sides by $-4c^2$.

$$\frac{\partial^2 u}{\partial \alpha \partial \beta} = 0$$

$$\frac{\partial}{\partial \alpha}\left(\frac{\partial u}{\partial \beta}\right) = 0$$

Solving for u is now very straightforward. Integrate both sides partially with respect to α.

$$\frac{\partial u}{\partial \beta} = g(\beta)$$

Here g is an arbitrary function. Integrate both sides partially with respect to β.

$$u(\alpha, \beta) = \int^{\beta} g(s)\, ds + F(\alpha) = G(\beta) + F(\alpha)$$

Now that u is known, change back to the original variables.

$$u(x, t) = G(x - ct) + F(x + ct)$$