Exercise 3

(a) Find a formula that describes the function in Figure 9.

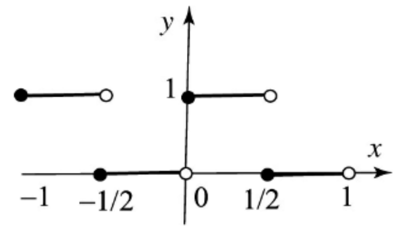

Figure 9 for Exercise 3.

(b) Describe the set of points where f is continuous. Compute $f(x+)$ and $f(x-)$ at all points x where f is not continuous. Is the function piecewise continuous?

(c) Compute $f'(x)$ at the points where the derivative exists. Compute $f'(x+)$ and $f'(x-)$ at the points where the derivative does not exist. Is the function piecewise smooth?

Solution

Part (a)

Notice that the function repeats itself every unit, so the period is 1.

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \le x < 1 \\ f(x+1) & \text{otherwise} \end{cases}$$

Part (b)

The function is continuous everywhere except at every half integer.

$$\left\{ x \;\middle|\; x \neq \frac{k}{2}, \quad k = 0, \pm 1, \pm 2, \ldots \right\}$$

From the left the limit of the function at the discontinuities is

$$\lim_{x \to \left(\frac{k}{2}\right)^-} f(x) = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases},$$

and from the right the limit of the function at the discontinuities is

$$\lim_{x \to \left(\frac{k}{2}\right)^+} f(x) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases}.$$

The function is piecewise continuous on $0 \le x \le 1$ because $f(0+)$ and $f(1-)$ exist, and there are only a finite number of discontinuities in $0 < x < 1$ that each have existing one-sided limits.

Part (c)

The derivative of f is undefined wherever f is discontinuous.

$$f'(x) = \begin{cases} \frac{d}{dx}(1) & \text{if } 0 < x < \frac{1}{2} \\ \frac{d}{dx}(0) & \text{if } \frac{1}{2} < x < 1 \\ f'(x+1) & \text{otherwise} \end{cases}$$

$$= \begin{cases} 0 & \text{if } 0 < x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x < 1 \\ f'(x+1) & \text{otherwise} \end{cases}$$

$$= 0, \quad x \neq \frac{k}{2}, \text{ where } k = 0, \pm 1, \pm 2, \ldots$$

Compute the limits of the derivative at the discontinuities.

$$\lim_{x \to \left(\frac{k}{2}\right)^-} f'(x) = 0$$

$$\lim_{x \to \left(\frac{k}{2}\right)^+} f'(x) = 0$$

The function's derivative is piecewise continuous on $0 \leq x \leq 1$ because $f'(0+)$ and $f'(1-)$ exist, and there are only a finite number of discontinuities in $0 < x < 1$ that each have existing one-sided limits. Therefore, f is piecewise smooth on $0 \leq x \leq 1$.