Exercise 5

Establish the orthogonality of the trigonometric system over the interval $[-\pi, \pi]$.

Solution

The aim here is to prove the orthogonality of the trigonometric functions on $-\pi \leq x \leq \pi$.

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad \text{if } m \neq n,$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0 \quad \text{for all } m \text{ and } n,$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0 \quad \text{if } m \neq n.$$

Prove the first identity, assuming that m and n are integers with $m \neq n$ and using the product-to-sum formulas for cosine-cosine.

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= \int_{-\pi}^{\pi} \frac{1}{2}[\cos(mx - nx) + \cos(mx + nx)] \, dx \\
&= \frac{1}{2}\left\{\int_{-\pi}^{\pi} \cos[(m-n)x] \, dx + \int_{-\pi}^{\pi} \cos[(m+n)x] \, dx\right\} \\
&= \frac{1}{2}\left\{ \frac{1}{m-n} \sin[(m-n)x]\bigg|_{-\pi}^{\pi} + \frac{1}{m+n} \sin[(m+n)x]\bigg|_{-\pi}^{\pi} \right\} \\
&= \frac{1}{2}\left\{ \frac{1}{m-n}\{\sin[(m-n)\pi] - \sin[-(m-n)\pi]\} \right. \\
&\qquad \left. + \frac{1}{m+n}\{\sin[(m+n)\pi] - \sin[-(m+n)\pi]\right\} \\
&= \frac{1}{2}\left\{ \frac{1}{m-n}\{\sin[(m-n)\pi] + \sin[(m-n)\pi]\} \right. \\
&\qquad \left. + \frac{1}{m+n}\{\sin[(m+n)\pi] + \sin[(m+n)\pi]\} \right\} \\
&= \frac{1}{2}\left\{ \frac{1}{m-n}\{2\sin[(m-n)\pi]\} + \frac{1}{m+n}\{2\sin[(m+n)\pi]\} \right\} \\
&= \frac{1}{m-n} \underbrace{\sin[(m-n)\pi]}_{=0} + \frac{1}{m+n} \underbrace{\sin[(m+n)\pi]}_{=0} \\
&= 0
\end{aligned}$$

$m - n$ and $m + n$ are integers, and the sine of an integer multiple of π is zero.

Prove the second identity, assuming that m and n are integers and using the product-to-sum formulas for cosine-sine.

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos mx \sin nx \, dx &= \int_{-\pi}^{\pi} \frac{1}{2}[\sin(mx+nx) - \sin(mx-nx)] \, dx \\
&= \frac{1}{2}\left\{\int_{-\pi}^{\pi} \sin[(m+n)x] \, dx - \int_{-\pi}^{\pi} \sin[(m-n)x] \, dx\right\} \\
&= \frac{1}{2}\left\{-\frac{1}{m+n}\cos[(m+n)x]\bigg|_{-\pi}^{\pi} + \frac{1}{m-n}\cos[(m-n)x]\bigg|_{-\pi}^{\pi}\right\} \\
&= \frac{1}{2}\left\{-\frac{1}{m+n}\{\cos[(m+n)\pi] - \cos[-(m+n)\pi]\}\right. \\
&\qquad\qquad \left. + \frac{1}{m-n}\{\cos[(m-n)\pi] - \cos[-(m-n)\pi]\right\} \\
&= \frac{1}{2}\left\{-\frac{1}{m+n}\{\cos[(m+n)\pi] - \cos[(m+n)\pi]\}\right. \\
&\qquad\qquad \left. + \frac{1}{m-n}\{\cos[(m-n)\pi] - \cos[(m-n)\pi]\right\} \\
&= \frac{1}{2}\left[-\frac{1}{m+n}(0) + \frac{1}{m-n}(0)\right] \\
&= 0
\end{aligned}$$

This is due to the fact that cosine is an even function: $\cos x = \cos(-x)$.

Prove the third identity, assuming that m and n are integers with $m \neq n$ and using the product-to-sum formulas for sine-sine.

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= \int_{-\pi}^{\pi} \frac{1}{2}[\cos(mx - nx) - \cos(mx + nx)] \, dx \\
&= \frac{1}{2} \left\{ \int_{-\pi}^{\pi} \cos[(m-n)x] \, dx - \int_{-\pi}^{\pi} \cos[(m+n)x] \, dx \right\} \\
&= \frac{1}{2} \left\{ \frac{1}{m-n} \sin[(m-n)x] \bigg|_{-\pi}^{\pi} - \frac{1}{m+n} \sin[(m+n)x] \bigg|_{-\pi}^{\pi} \right\} \\
&= \frac{1}{2} \left\{ \frac{1}{m-n} \{\sin[(m-n)\pi] - \sin[-(m-n)\pi]\} \right. \\
&\qquad \left. - \frac{1}{m+n} \{\sin[(m+n)\pi] - \sin[-(m+n)\pi]\} \right\} \\
&= \frac{1}{2} \left\{ \frac{1}{m-n} \{\sin[(m-n)\pi] + \sin[(m-n)\pi]\} \right. \\
&\qquad \left. - \frac{1}{m+n} \{\sin[(m+n)\pi] + \sin[(m+n)\pi]\} \right\} \\
&= \frac{1}{2} \left\{ \frac{1}{m-n} \{2\sin[(m-n)\pi]\} - \frac{1}{m+n} \{2\sin[(m+n)\pi]\} \right\} \\
&= \frac{1}{m-n} \underbrace{\sin[(m-n)\pi]}_{=0} - \frac{1}{m+n} \underbrace{\sin[(m+n)\pi]}_{=0} \\
&= 0
\end{aligned}$$

$m - n$ and $m + n$ are integers, and the sine of an integer multiple of π is zero.