Problem 1.21

By reduction of order, find the general solution of $x^2y'' - 4xy' + 6y = x^4 \sin x$ after observing that $y_1 = x^2$ is a solution of the associated homogeneous equation.

Solution

With the solution to the associated homogeneous equation in hand, we can find the general solution with reduction of order—also known as multiplicative substitution.

$$y(x) = y_1 u(x) = x^2 u(x) \tag{1}$$

Find the derivatives of y in terms of the new variable u.

$$\begin{aligned} y'(x) &= 2xu(x) + x^2u'(x) \\ y''(x) &= 2u(x) + 2xu'(x) + 2xu'(x) + x^2u''(x) = 2u(x) + 4xu'(x) + x^2u''(x) \end{aligned}$$

Plug these expressions into the ODE now.

$$x^2(2u + 4xu' + x^2u'') - 4x(2xu + x^2u') + 6x^2u = x^4 \sin x$$

Expand the terms on the left side.

$$2x^2u + 4x^3u' + x^4u'' - 8x^2u - 4x^3u' + 6x^2u = x^4 \sin x$$

Combine like-terms.

$$x^4u'' = x^4 \sin x$$

Divide both sides by x^4.

$$u'' = \sin x$$

Integrate both sides with respect to x.

$$u' = -\cos x + C_1$$

Integrate both sides with respect to x again.

$$u(x) = -\sin x + C_1x + C_2$$

Now that we have $u(x)$, we can obtain $y(x)$ by multiplying the result by x^2 as equation (1) indicates.

$$y(x) = -x^2 \sin x + C_1x^3 + C_2x^2$$

This is the general solution to the ODE. Note that it includes the solution, $y_1 = x^2$, we started out with.