Problem 1.29

Find a differential equation having the general solution $y = c_1(x + c_2)^n$.

Solution

There are two constants of integration, c_1 and c_2, so we expect the differential equation to be second order. Take two derivatives of the solution.

$$\begin{aligned} y &= c_1(x + c_2)^n \\ y' &= c_1 n(x + c_2)^{n-1} = \frac{c_1 n(x + c_2)^n}{x + c_2} \\ y'' &= c_1 n(n-1)(x + c_2)^{n-2} = \frac{c_1 n(n-1)(x + c_2)^n}{(x + c_2)^2} \end{aligned}$$

The equation for y'' will ultimately be our differential equation. Use the equations for y and y' to eliminate c_1 and c_2 with substitution. We'll use y to eliminate c_1 first.

$$\begin{aligned} y' &= \frac{ny}{x + c_2} \\ y'' &= \frac{n(n-1)y}{(x + c_2)^2} \end{aligned}$$

Now use y' to eliminate c_2.

$$y' = \frac{ny}{x + c_2} \quad \rightarrow \quad \frac{y'}{ny} = \frac{1}{x + c_2} \quad \rightarrow \quad \left(\frac{y'}{ny}\right)^2 = \frac{1}{(x + c_2)^2}$$

So we have

$$y'' = n(n-1)y \cdot \frac{(y')^2}{n^2 y^2}.$$

Therefore, the differential equation that has the general solution, $y = c_1(x + c_2)^n$, is

$$y'' = \frac{n-1}{n} \cdot \frac{(y')^2}{y}.$$