Problem 1.30

(a) At $t = 0$, a pig, initially at the origin, runs along the x axis with constant speed v. At $t = 0$, a farmer, initially 20 yd north of the origin, also runs with constant speed v. If the farmer's instantaneous velocity is always directed toward the instantaneous position of the pig, show that the farmer never gets closer than 10 yd from the pig.

(b) Now suppose that the pig starts over again from $x = 0$, $y = 0$ at $t = 0$ and starts running with the speed v. The farmer still starts 20 yd north of the pig but can now run at a speed of $\frac{3}{2}v$. The farmer is assisted by his daughter who starts 15 yd south of the pig at $t = 0$ and can run at a speed of $\frac{4}{3}v$. If both the farmer and the farmer's daughter always run toward the instantaneous position of the pig, who catches the pig first?

(c) At $t = 0$, a pig initially at $(1, 0)$ starts to run around the unit circle with constant speed v. At $t = 0$, a farmer initially at the origin runs with constant speed v and instantaneous velocity directed toward the instantaneous position of the pig. Does the farmer catch the pig?

Solution

Part (a)

I prefer to use a different coordinate system than the one Bender & Orszag have set up for us. Let the positive y-axis point to the east and let the positive x-axis point to the north.

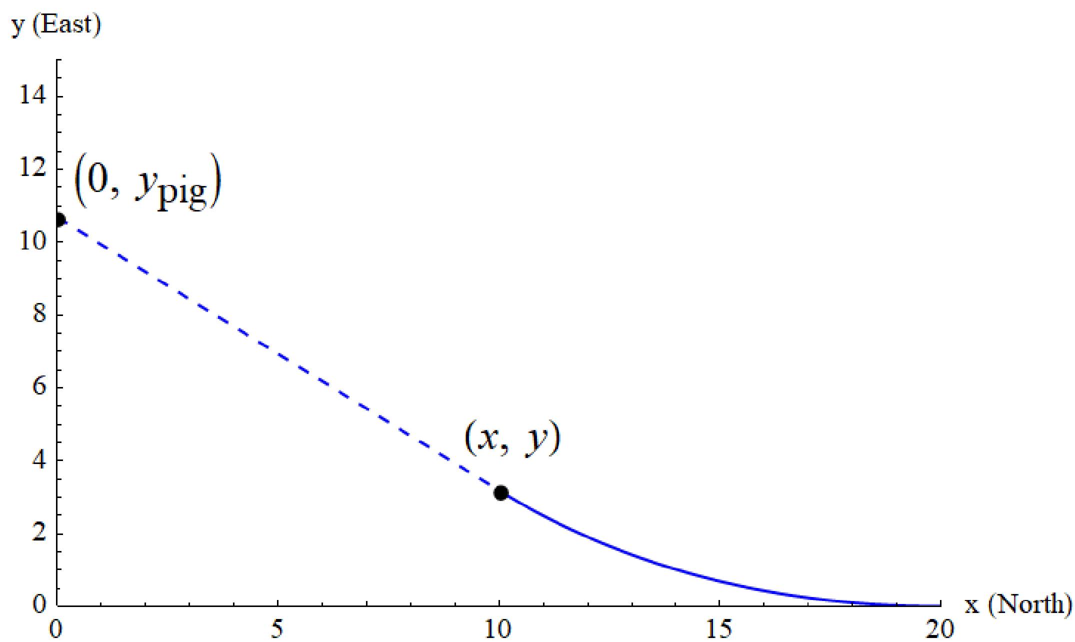

Figure 1: This figure illustrates the new coordinate system as well as the farmer's path in it. All units are in yards.

Initially the farmer is standing at $(20, 0)$ and is facing towards the pig's location, the origin. These facts tell us the boundary conditions of the differential equation that governs the farmer's motion. Let the farmer's path be represented as $y(x)$.

$$y(20) = 0 \tag{1}$$

$$y'(20) = 0 \tag{2}$$

Our task now is to derive the ODE for the farmer's path from first principles. If we have two points in the xy-plane, (x_1, y_1) and (x_2, y_2), then the slope of the line passing through these two points is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope of the farmer's path at x is $y'(x)$, the first point is (x, y), and the second point is $(0, y_{\text{pig}})$. So we have

$$\frac{dy}{dx} = \frac{y_{\text{pig}} - y}{0 - x}$$

for the equation of motion. The next task is to write y_{pig} in terms of x and y. Because the farmer and the pig travel at the same speed, that means they cover the same distance in the same amount of time. y_{pig} is therefore equal to the arc length of the farmer's path.

$$y_{\text{pig}} = \int_x^{20} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Plugging this expression into the ODE gives us

$$\frac{dy}{dx} = \frac{\int_x^{20} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx - y}{0 - x}.$$

Multiply both sides by x.

$$x\frac{dy}{dx} = y - \int_x^{20} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Move the variable x to the upper limit by switching the sign in front of the integral.

$$x\frac{dy}{dx} = y + \int_{20}^{x} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

To wipe out the integral from the equation, differentiate both sides with respect to x.

$$\cancel{\frac{dy}{dx}} + x\frac{d^2y}{dx^2} = \cancel{\frac{dy}{dx}} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

The ODE for the farmer's path is thus

$$x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

The ODE is first-order in y', so make the substitution

$$u = \frac{dy}{dx}$$

$$\frac{du}{dx} = \frac{d^2y}{dx^2}.$$

Plug these expressions into the ODE.

$$x\frac{du}{dx} = \sqrt{1+u^2}$$

This can be solved with separation of variables.

$$\frac{du}{\sqrt{1+u^2}} = \frac{dx}{x}$$

Integrate both sides.

$$\int^u \frac{ds}{\sqrt{1+s^2}} = \ln|x| + C$$

Use trigonometric substitution to solve the integral on the left side.

$$\begin{aligned} s &= \tan\theta \quad \rightarrow \quad 1 + s^2 = 1 + \tan^2\theta = \sec^2\theta \\ ds &= \sec^2\theta\, d\theta \end{aligned}$$

Substitute these expressions into the integrand. Don't forget to change the upper limit as well.

$$\int^{\tan^{-1}u} \frac{\sec^2\theta\, d\theta}{\sqrt{\sec^2\theta}} = \ln|x| + C$$

The integral simplifies to

$$\int^{\tan^{-1}u} \sec\theta\, d\theta = \ln|x| + C.$$

Evaluate the integral.

$$\ln|\sec\theta + \tan\theta|\Big|^{\tan^{-1}u} = \ln|x| + C$$

Plug in the upper limit for θ.

$$\ln|\sec\tan^{-1}u + \tan\tan^{-1}u| = \ln|x| + C$$

Draw out the right triangle that is implied from $\theta = \tan^{-1}u$, or $\tan\theta = u$.

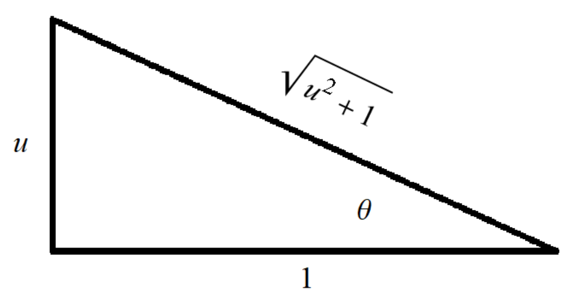

Figure 2: The right triangle.

From this triangle, we have the following relations.

$$\begin{aligned} \sec\tan^{-1}u &= \sqrt{u^2+1} \\ \tan\tan^{-1}u &= u \end{aligned}$$

So we have

$$\ln|\sqrt{u^2+1}+u| = \ln|x| + C.$$

Exponentiate both sides.

$$|\sqrt{u^2+1}+u| = |x|e^C$$

Introduce $\pm$ on the right side in order to remove the absolute value sign on the left.

$$\sqrt{u^2+1}+u = \pm e^C|x|$$

Use a new arbitrary constant.

$$\sqrt{u^2+1}+u = A|x|$$

Since the farmer always stays in the first quadrant, $x > 0$, so the absolute value sign on x can be dropped.

$$\sqrt{u^2+1}+u = Ax$$

The aim now is to solve for u. Bring u to the right side.

$$\sqrt{u^2+1} = Ax - u$$

Square both sides.

$$\cancel{u^2} + 1 = A^2x^2 - 2Axu + \cancel{u^2}$$

Bring $2Axu$ to the left side and 1 to the right.

$$2Axu = A^2x^2 - 1$$

Divide both sides by $2Ax$.

$$u(x) = \frac{Ax}{2} - \frac{1}{2Ax}$$

Now that u is solved for, change back to the original variable y.

$$\frac{dy}{dx} = \frac{Ax}{2} - \frac{1}{2Ax}$$

At this point, we can use the second boundary condition, equation (2), to determine A.

$$0 = 10A - \frac{1}{40A}$$

Solving for A gives

$$A = \pm\frac{1}{20}.$$

We choose the positive value for A so that dy/dx is negative for $0 < x < 20$.

$$\frac{dy}{dx} = \frac{x}{40} - \frac{10}{x}$$

Integrate both sides with respect to x to solve for y.

$$y(x) = \frac{x^2}{80} - 10\ln x + B$$

Apply the first boundary condition, equation (1), to determine B.

$$0 = \frac{20^2}{80} - 10\ln 20 + B$$

Solving for B gives

$$B = 10\ln 20 - 5.$$

Plugging this into the equation for y, we have

$$y(x) = \frac{x^2}{80} - 10\ln x + 10\ln 20 - 5.$$

Therefore, the farmer's path is

$$y(x) = \frac{x^2}{80} + 10\ln\frac{20}{x} - 5.$$

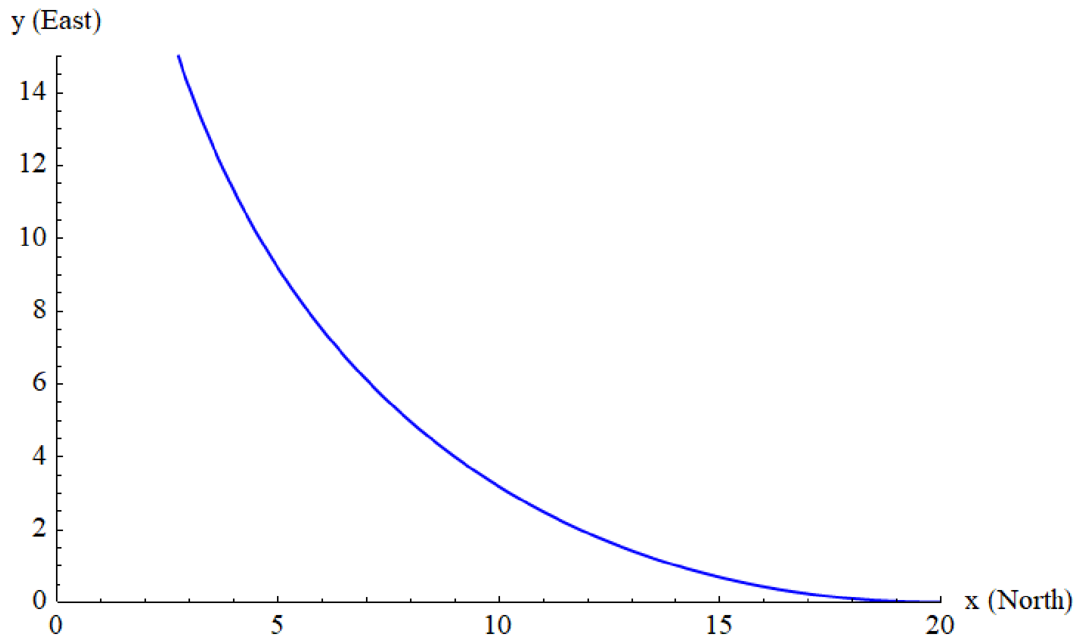

Figure 3: The farmer's path. All units are in yards.

Looking at Figure 1, we can get the distance between the farmer and the pig by finding the tangent line, $y = y_T(x)$, to the farmer's path at some point, $x = c$. The arc length of this line from 0 to c will be the distance we're looking for. The strategy will be to find the value of c that minimizes the arc length of the tangent line. Use the point-slope formula to get the equation of the tangent line. One point is on the tangent line, $(x, y_T(x))$, and the other point is on the farmer's path, $(c, y(c))$.

$$y_T(x) - y(c) = y'(c)(x - c)$$

Move $y(c)$ to the right side.

$$y_T(x) = y'(c)(x - c) + y(c)$$

As mentioned before, the distance from the pig to the farmer is the arc length of $y_T(x)$ from 0 to c.

$$d(c) = \int_0^c \sqrt{1 + (y_T')^2}\,dx = \int_0^c \sqrt{1 + [y'(c)]^2}\,dx = c\sqrt{1 + [y'(c)]^2} = c\sqrt{1 + \left(\frac{10}{c} - \frac{c}{40}\right)^2}$$

To minimize this function, take the derivative of it.

$$d'(c) = \sqrt{1+\left(\frac{10}{c}-\frac{c}{40}\right)^2} + c\cdot\frac{1}{2}\left[1+\left(\frac{10}{c}-\frac{c}{40}\right)^2\right]^{-1/2}\cdot 2\left(\frac{10}{c}-\frac{c}{40}\right)\cdot\left(-\frac{10}{c^2}-\frac{1}{40}\right)$$

Expanding the expression and combining the like-terms leaves us with a simple expression.

$$d'(c) = \frac{c}{2}$$

$d'(c) = 0$ only occurs when $c = 0$, which means the farmer is closest to the pig when he gets to the y-axis. To find the minimum distance between him and the pig, take the limit as $c \to 0$ of the distance function.

$$\begin{aligned}
\lim_{c\to 0} d(c) = \lim_{c\to 0} c\sqrt{1+\left(\frac{10}{c}-\frac{c}{40}\right)^2} &= \lim_{c\to 0}\sqrt{c^2+c^2\left(\frac{10}{c}-\frac{c}{40}\right)^2} \\
&= \lim_{c\to 0}\sqrt{c^2+\left[c\left(\frac{10}{c}-\frac{c}{40}\right)\right]^2} \\
&= \lim_{c\to 0}\sqrt{\underbrace{c^2}_{=\,0}+\left(10-\underbrace{\frac{c^2}{40}}_{=\,0}\right)^2} \\
&= \sqrt{10^2} \\
&= 10
\end{aligned}$$

Therefore, the farmer never gets closer than 10 yards from the pig.

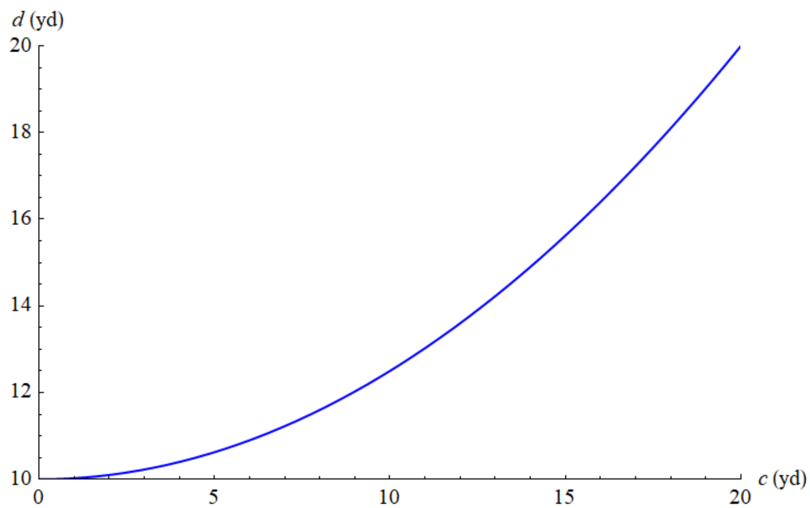

Figure 4: Plot of the farmer's distance from the pig as a function of his distance from the y-axis. When he's 20 yards away from the y-axis initially, he's 20 yards away from the pig. If he runs directly toward the pig indefinitely, then his distance to the pig when he reaches the y-axis is 10 yards.

Part (b)

We will assume that the pig runs to the east as in part (a). Let the positive y-axis point to the east and let the positive x-axis point to the north. Because there are two people now, the farmer and his daughter, we will have to derive two equations of motion.

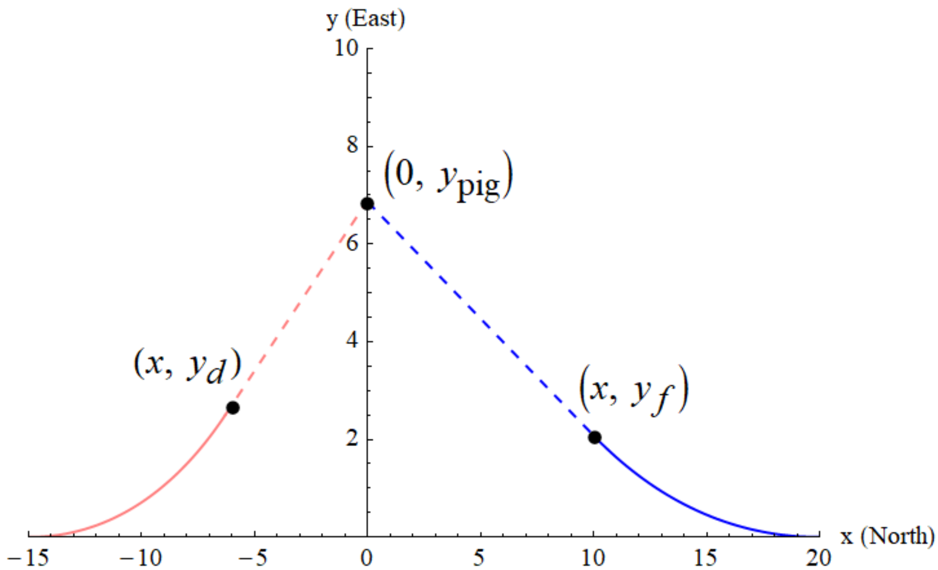

Figure 5: This figure illustrates the coordinate system as well as the paths of the farmer and daughter in it. All units are in yards.

The Farmer's Equation of Motion

Initially the farmer is standing at $(20, 0)$ and is facing towards the pig's location, the origin. These facts tell us the boundary conditions of the differential equation that governs the farmer's motion.

$$y_f(20) = 0 \quad (1)$$
$$y_f'(20) = 0 \quad (2)$$

Our task now is to derive the ODE for the farmer's path from first principles. If we have two points in the xy-plane, (x_1, y_1) and (x_2, y_2), then the slope of the line passing through these two points is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope of the farmer's path at x is $y_f'(x)$, the first point is (x, y_f), and the second point is $(0, y_{\text{pig}})$. So we have

$$\frac{dy_f}{dx} = \frac{y_{\text{pig}} - y_f}{0 - x}$$

for the equation of motion. The next task is to write y_{pig} in terms of x and y_f. Because the farmer travels at 1.5 times the speed of the pig, that means he covers 1.5 times the distance of the pig in the same amount of time. $1.5y_{\text{pig}}$ is therefore equal to the arc length of the farmer's path.

$$\frac{3}{2}y_{\text{pig}} = \int_x^{20} \sqrt{1 + \left(\frac{dy_f}{dx}\right)^2}\, dx \quad \rightarrow \quad y_{\text{pig}} = \frac{2}{3}\int_x^{20} \sqrt{1 + \left(\frac{dy_f}{dx}\right)^2}\, dx$$

Plugging this expression into the ODE gives us

$$\frac{dy_f}{dx} = \frac{\frac{2}{3}\int_x^{20}\sqrt{1+\left(\frac{dy_f}{dx}\right)^2}\,dx - y_f}{0-x}.$$

Multiply both sides by x.

$$x\frac{dy_f}{dx} = y_f - \frac{2}{3}\int_x^{20}\sqrt{1+\left(\frac{dy_f}{dx}\right)^2}\,dx$$

Move the variable x to the upper limit by switching the sign in front of the integral.

$$x\frac{dy_f}{dx} = y_f + \frac{2}{3}\int_{20}^{x}\sqrt{1+\left(\frac{dy_f}{dx}\right)^2}\,dx$$

To wipe out the integral from the equation, differentiate both sides with respect to x.

$$\cancel{\frac{dy_f}{dx}} + x\frac{d^2y_f}{dx^2} = \cancel{\frac{dy_f}{dx}} + \frac{2}{3}\sqrt{1+\left(\frac{dy_f}{dx}\right)^2}$$

The ODE for the farmer's path is thus

$$x\frac{d^2y_f}{dx^2} = \frac{2}{3}\sqrt{1+\left(\frac{dy_f}{dx}\right)^2}.$$

The ODE is first-order in y_f', so make the substitution

$$u = \frac{dy_f}{dx}$$
$$\frac{du}{dx} = \frac{d^2y_f}{dx^2}.$$

Plug these expressions into the ODE.

$$x\frac{du}{dx} = \frac{2}{3}\sqrt{1+u^2}$$

This can be solved with separation of variables.

$$\frac{du}{\sqrt{1+u^2}} = \frac{2}{3}\frac{dx}{x}$$

Integrate both sides.

$$\int^u \frac{ds}{\sqrt{1+s^2}} = \frac{2}{3}\ln|x| + C$$

Use trigonometric substitution to solve the integral on the left side.

$$s = \tan\theta \quad \rightarrow \quad 1+s^2 = 1+\tan^2\theta = \sec^2\theta$$
$$ds = \sec^2\theta\,d\theta$$

Substitute these expressions into the integrand. Don't forget to change the upper limit as well.

$$\int^{\tan^{-1}u} \frac{\sec^2\theta\,d\theta}{\sqrt{\sec^2\theta}} = \frac{2}{3}\ln|x| + C$$

The integral simplifies to

$$\int^{\tan^{-1} u} \sec\theta\, d\theta = \frac{2}{3}\ln|x| + C.$$

Evaluate the integral.

$$\ln|\sec\theta + \tan\theta|\Big|^{\tan^{-1} u} = \frac{2}{3}\ln|x| + C$$

Plug in the upper limit for θ.

$$\ln|\sec\tan^{-1} u + \tan\tan^{-1} u| = \frac{2}{3}\ln|x| + C$$

Draw out the right triangle that is implied from $\theta = \tan^{-1} u$, or $\tan\theta = u$.

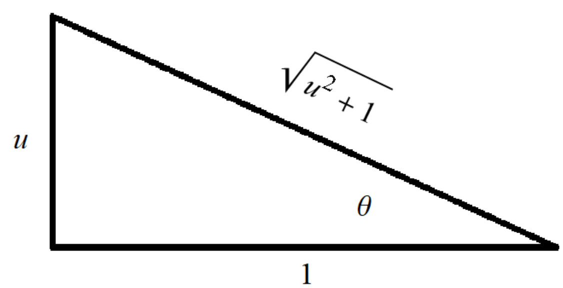

Figure 6: The right triangle.

From this triangle, we have the following relations.

$$\sec\tan^{-1} u = \sqrt{u^2+1}$$
$$\tan\tan^{-1} u = u$$

So we have

$$\ln|\sqrt{u^2+1} + u| = \frac{2}{3}\ln|x| + C.$$

Bring the coefficient of the logarithm to the argument's exponent and exponentiate both sides.

$$|\sqrt{u^2+1} + u| = |x|^{2/3}e^C$$

Introduce $\pm$ on the right side in order to remove the absolute value sign on the left.

$$\sqrt{u^2+1} + u = \pm e^C|x|^{2/3}$$

Use a new arbitrary constant.

$$\sqrt{u^2+1} + u = A|x|^{2/3}$$

Since the farmer always stays in the first quadrant, $x > 0$, so the absolute value sign on x can be dropped.

$$\sqrt{u^2+1} + u = Ax^{2/3}$$

The aim now is to solve for u. Bring u to the right side.

$$\sqrt{u^2+1} = Ax^{2/3} - u$$

Square both sides.

$$\cancel{u^2} + 1 = A^2x^{4/3} - 2Ax^{2/3}u + \cancel{u^2}$$

Bring $2Ax^{2/3}u$ to the left side and 1 to the right.

$$2Ax^{2/3}u = A^2x^{4/3} - 1$$

Divide both sides by $2Ax^{2/3}$.

$$u(x) = \frac{Ax^{2/3}}{2} - \frac{1}{2Ax^{2/3}}$$

Now that u is solved for, change back to the original variable y_f.

$$\frac{dy_f}{dx} = \frac{Ax^{2/3}}{2} - \frac{1}{2Ax^{2/3}}$$

At this point, we can use the second boundary condition, equation (2), to determine A.

$$0 = \frac{A \cdot 20^{2/3}}{2} - \frac{1}{2A \cdot 20^{2/3}}$$

Solving for A gives

$$A = \pm\frac{1}{2\sqrt[3]{50}}.$$

We choose the positive value for A so that dy_f/dx is negative for $0 < x < 20$.

$$\frac{dy_f}{dx} = \frac{x^{2/3}}{4\sqrt[3]{50}} - \frac{\sqrt[3]{50}}{x^{2/3}}$$

Integrate both sides with respect to x to solve for y_f.

$$y_f(x) = \frac{3x^{5/3}}{20\sqrt[3]{50}} - 3x^{1/3}\sqrt[3]{50} + B$$

Apply the first boundary condition, equation (1), to determine B.

$$0 = \frac{3 \cdot 20^{5/3}}{20\sqrt[3]{50}} - 3 \cdot 20^{1/3}\sqrt[3]{50} + B$$

Solving for B gives

$$B = 24.$$

Plugging this into the equation for y, we therefore have

$$y_f(x) = \frac{3}{20}\sqrt[3]{\frac{x^5}{50}} - 3\sqrt[3]{50x} + 24$$

for the farmer's path.

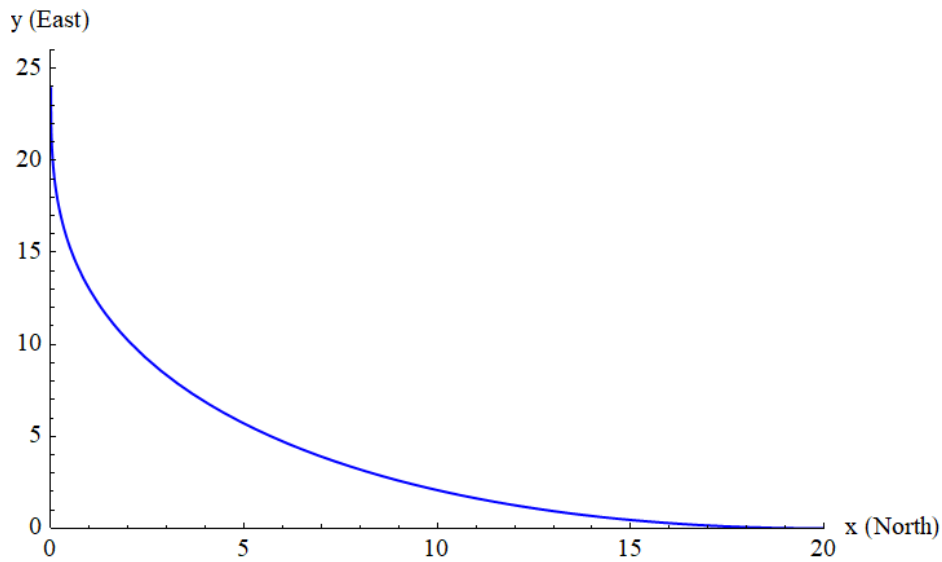

Figure 7: The farmer's path. All units are in yards.

The Daughter's Equation of Motion

Initially the daughter is standing at $(-15, 0)$ and is facing towards the pig's location, the origin. These facts tell us the boundary conditions of the differential equation that governs the daughter's motion.

$$y_d(-15) = 0 \tag{3}$$

$$y_d'(-15) = 0 \tag{4}$$

Our task now is to derive the ODE for the daughter's path from first principles. If we have two points in the xy-plane, (x_1, y_1) and (x_2, y_2), then the slope of the line passing through these two points is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope of the daughter's path at x is $y_d'(x)$, the first point is (x, y_d), and the second point is $(0, y_{\text{pig}})$. So we have

$$\frac{dy_d}{dx} = \frac{y_{\text{pig}} - y_d}{0 - x}$$

for the equation of motion. The next task is to write y_{pig} in terms of x and y_d. Because the daughter travels at 4/3 times the speed of the pig, that means she covers 4/3 times the distance of the pig in the same amount of time. $(4/3)y_{\text{pig}}$ is therefore equal to the arc length of the daughter's path.

$$\frac{4}{3}y_{\text{pig}} = \int_{-15}^{x} \sqrt{1 + \left(\frac{dy_d}{dx}\right)^2}\,dx \quad \rightarrow \quad y_{\text{pig}} = \frac{3}{4}\int_{-15}^{x} \sqrt{1 + \left(\frac{dy_d}{dx}\right)^2}\,dx$$

Plugging this expression into the ODE gives us

$$\frac{dy_d}{dx} = \frac{\frac{3}{4}\int_{-15}^{x} \sqrt{1 + \left(\frac{dy_d}{dx}\right)^2}\,dx - y_d}{0 - x}.$$

Multiply both sides by x.

$$x\frac{dy_d}{dx} = y_d - \frac{3}{4}\int_{-15}^{x}\sqrt{1+\left(\frac{dy_d}{dx}\right)^2}\,dx$$

To wipe out the integral from the equation, differentiate both sides with respect to x.

$$\cancel{\frac{dy_d}{dx}} + x\frac{d^2y_d}{dx^2} = \cancel{\frac{dy_d}{dx}} - \frac{3}{4}\sqrt{1+\left(\frac{dy_d}{dx}\right)^2}$$

The ODE for the daughter's path is thus

$$x\frac{d^2y_d}{dx^2} = -\frac{3}{4}\sqrt{1+\left(\frac{dy_d}{dx}\right)^2}.$$

The ODE is first-order in y_d', so make the substitution

$$u = \frac{dy_d}{dx}$$
$$\frac{du}{dx} = \frac{d^2y_d}{dx^2}.$$

Plug these expressions into the ODE.

$$x\frac{du}{dx} = -\frac{3}{4}\sqrt{1+u^2}$$

This can be solved with separation of variables.

$$\frac{du}{\sqrt{1+u^2}} = -\frac{3}{4}\frac{dx}{x}$$

Integrate both sides.

$$\ln|\sqrt{u^2+1}+u| = -\frac{3}{4}\ln|x| + C$$

Bring the coefficient of the logarithm to the argument's exponent and exponentiate both sides.

$$|\sqrt{u^2+1}+u| = |x|^{-3/4}e^C$$

Introduce $\pm$ on the right side in order to remove the absolute value sign on the left.

$$\sqrt{u^2+1}+u = \pm e^C|x|^{-3/4}$$

Use a new arbitrary constant.

$$\sqrt{u^2+1}+u = E|x|^{-3/4}$$

Since the daughter always stays in the second quadrant, $x < 0$, so the absolute value sign on x can be dropped as long as we put a minus sign in front of x.

$$\sqrt{u^2+1}+u = E(-x)^{-3/4}$$

The aim now is to solve for u. Bring u to the right side.

$$\sqrt{u^2+1} = E(-x)^{-3/4} - u$$

Square both sides.

$$\cancel{u^2} + 1 = E^2(-x)^{-3/2} - 2E(-x)^{-3/4}u + \cancel{u^2}$$

Bring $2E(-x)^{-3/4}u$ to the left side and 1 to the right.

$$2E(-x)^{-3/4}u = E^2(-x)^{-3/2} - 1$$

Divide both sides by $2E(-x)^{-3/4}$.

$$u(x) = \frac{E(-x)^{-3/4}}{2} - \frac{(-x)^{3/4}}{2E}$$

Now that u is solved for, change back to the original variable y_d.

$$\frac{dy_d}{dx} = \frac{E(-x)^{-3/4}}{2} - \frac{(-x)^{3/4}}{2E}$$

At this point, we can use the second boundary condition, equation (4), to determine E.

$$0 = \frac{E(15)^{-3/4}}{2} - \frac{(15)^{3/4}}{2E}$$

Solving for E gives

$$E = \pm 15^{3/4}.$$

We choose the positive value for E so that dy_d/dx is positive for $-15 < x < 0$.

$$\frac{dy_d}{dx} = \frac{15^{3/4}(-x)^{-3/4}}{2} - \frac{(-x)^{3/4}}{2 \cdot 15^{3/4}}$$

Integrate both sides with respect to x to solve for y_d.

$$y_d(x) = -\frac{4 \cdot 15^{3/4}(-x)^{1/4}}{2} + \frac{4}{7}\frac{(-x)^{7/4}}{2 \cdot 15^{3/4}} + F$$

Apply the first boundary condition, equation (3), to determine F.

$$0 = -2 \cdot 15^{3/4}(15)^{1/4} + \frac{2}{7}\frac{(15)^{7/4}}{15^{3/4}} + F$$

Solving for F gives

$$F = \frac{180}{7}.$$

Plugging this into the equation for y, we therefore have

$$y_d(x) = -2 \cdot 15^{3/4}(-x)^{1/4} + \frac{2}{7}\frac{(-x)^{7/4}}{15^{3/4}} + \frac{180}{7}$$

for the daughter's path.

To see whether the farmer or the daughter catches the pig first, we have to compare the y-intercepts of their paths.

$$y_f(0) = 24$$
$$y_d(0) = \frac{180}{7} \approx 25.71$$

The farmer catches the pig when it is only 24 yards to the east, whereas the daughter would catch the pig when it is about 25.71 yards to the east. Therefore, the farmer catches the pig first despite his being further away at the start.

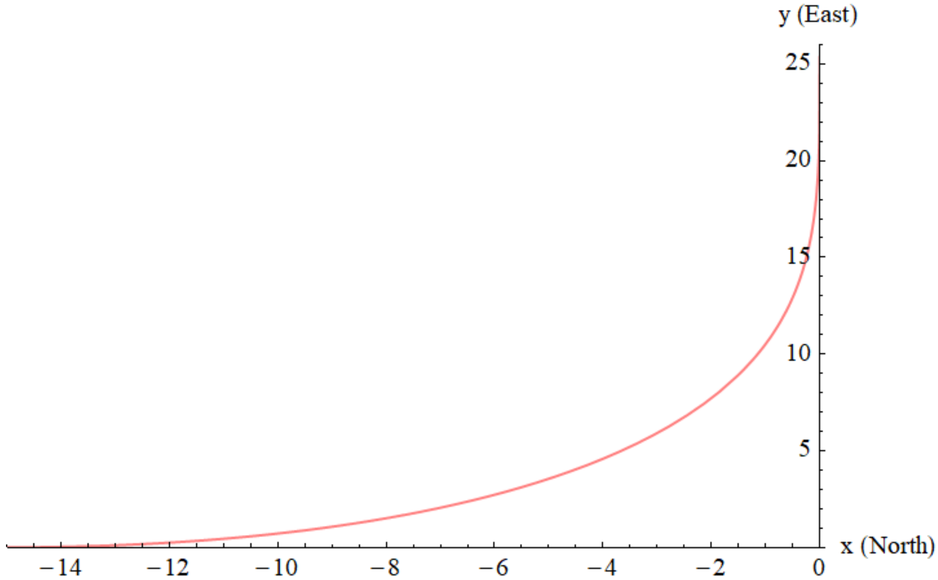

Figure 8: The daughter's path. All units are in yards.

Part (c)

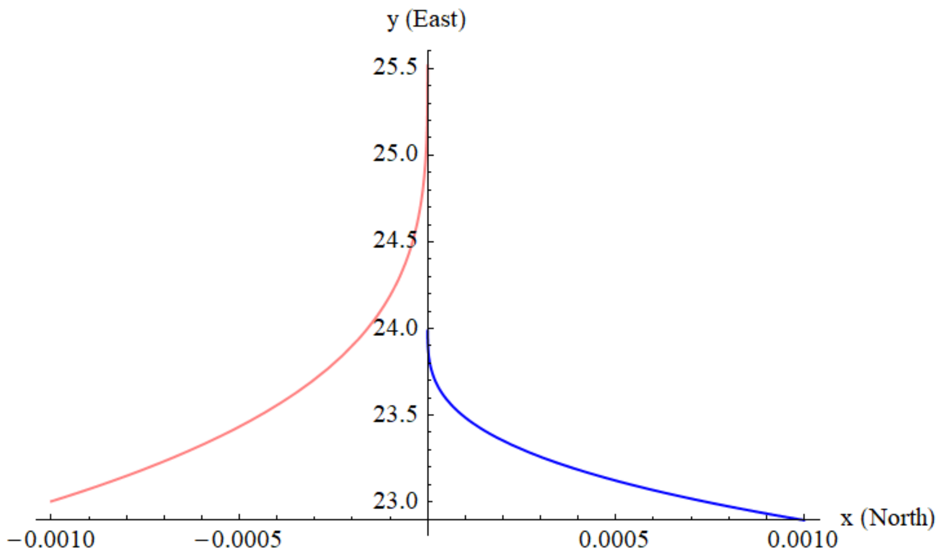

Figure 9: The farmer and daughter's paths zoomed in towards the end. All units are in yards. The farmer catches the pig when it reaches the 24th yard. The daughter would catch the pig only after it passed the 25th yard.